Received: *October 11, 1999,* Accepted: *December 17, 1999*

Dielectric-branes

Robert C. Myers

Institute for Theoretical Physics, University of California
Santa Barbara, CA 93106 USA, and
Department of Physics, McGill University
Montréal, QC, H3A 2T8, Canada[*]
E-mail: `rcm@hep.physics.mcgill.ca`

Abstract: We extend the usual world-volume action for a Dp-brane to the case of N coincident Dp-branes where the world-volume theory involves a U(N) gauge theory. The guiding principle in our construction is that the action should be consistent with the familiar rules of T-duality. The resulting action involves a variety of potential terms, i.e. nonderivative interactions, for the nonabelian scalar fields. This action also shows that Dp-branes naturally couple to RR potentials of all form degrees, including both larger and smaller than $p+1$. We consider the dynamics resulting from this action for Dp-branes moving in nontrivial background fields, and illustrate how the Dp-branes are "polarized" by external fields. In a simple example, we show that a system of D0-branes in an external RR four-form field expands into a noncommutative two-sphere, which is interpreted as the formation of a spherical D2-D0 bound state.

Keywords: D-branes, Brane Dynamics in Gauge Theories.

[*]Permanent address.

JHEP12(1999)022

Contents

1. Introduction

Our understanding of string theory has been transformed since 1995 [1]. One important aspect in this transformation was the appreciation of the important role of extended objects beyond strings in these theories. Of particular interest for the type-II (and I) superstring theories are the Dirichlet-branes (D-branes) which can be regarded as stringy solitons which carry Ramond-Ramond (RR) charges [2] — see also [3].

These objects arise in considering T-duality transformations of a ten-dimensional type-I superstring theory [4, 5, 6]. So as originally conceived, a Dp-brane is a $(p+1)$-dimensional extended surface in spacetime which supports the endpoints of open strings. The massless modes of this open string theory form a supersymmetric U(1) gauge theory with a vector A_a, $9-p$ real scalars Φ^i and their superpartner fermions. At leading order, the low-energy action corresponds to the dimensional reduction of that for ten-dimensional U(1) super Yang-Mills theory. As usual in string theory, there are higher order $\alpha' = \ell_s^2$ corrections, where ℓ_s is the string length scale. Following the work of ref. [7], Leigh showed that the action incorporating these corrections to all orders in the field strength takes the Born-Infeld form [8]

$$S_{BI} = -T_p \int d^{p+1}\sigma \;\left(e^{-\phi}\sqrt{-\det(P[G+B]_{ab} + 2\pi\ell_s^2\, F_{ab})}\right), \tag{1.1}$$

where T_p is the brane tension. The higher order corrections in this action can be trusted as long as derivatives of the gauge field strength (and second derivatives of the scalars) are small on the string scale ℓ_s. Leigh's action (1.1) also reveals that the D-branes are dynamical objects, whose transverse displacements[1] are described by Φ^i, i.e. $\Delta X^i(\sigma) = 2\pi\ell_s^2\Phi^i$. This dynamics is implicit in the pull-back of the bulk spacetime tensors to the D-brane world-volume, which is denoted by the symbol $P[\ldots]$, e.g.,

$$P[G]_{ab} = G_{ab} + 4\pi\ell_s^2\, G_{i(a}\,\partial_{b)}\Phi^i + 4\pi^2\ell_s^4\, G_{ij}\partial_a\Phi^i\partial_b\Phi^j\,. \tag{1.2}$$

Eq. (1.1) incorporates the couplings of the world-volume vector and scalars to the massless Neveu-Schwarz fields of the bulk closed string theory, i.e. the metric, dilaton and Kalb-Ramond field. Since a D-brane also carries an RR charge [2], there must be couplings to the massless RR states of the closed string. These interactions are incorporated in a Chern-Simons action of the form [9, 10]

$$S_{CS} = \mu_p \int P\left[\sum C^{(n)}\, e^B\right] e^{2\pi\ell_s^2\, F}\,, \tag{1.3}$$

where $C^{(n)}$ denote the n-form RR potentials and μ_p is the RR charge of the brane. Thus a Dp-brane is naturally charged under the $(p+1)$-form RR potential. However in the presence of background Kalb-Ramond fields or world-volume gauge fields, it may also carry a charge of the RR potentials with a lower form degree, as allowed with the couplings induced by the exponential factor[9]. Such configurations have an interpretation in terms of bound states of D-branes of different dimensions[11]. More exotic gauge field configurations also have interesting interpretations in terms of intersecting branes[12]. Note that in general the background spacetime fields are functions of the transverse coordinates, and so in the above actions (1.1) and (1.3), they become functionals of the scalars Φ^i. Hence, even in the leading low energy approximation, these scalars would be governed by a non-linear sigma model action.

One of the most remarkable properties of D-branes is that the U(1) gauge symmetry of an individual D-brane is enhanced to a nonabelian U(N) symmetry for N coincident D-branes [11]. As N parallel D-branes approach each other, the ground state modes of strings stretching between the D-branes become massless. These extra massless states carry the appropriate charges to then fill out U(N) representations and the $\mathrm{U(1)^N}$ of the individual D-branes is enhanced to U(N). Hence A_a becomes a nonabelian gauge field and the scalars Φ^i become scalars in the adjoint representation of U(N). Understanding how to accommodate this simple yet remarkable modification in the world-volume actions for general backgrounds has only received limited attention [13, 14].

The naive extension of the Chern-Simons action (1.3) is apparently straightforward. One would include an additional trace over gauge indices of the nonabelian

[1] Here and in the following equation, we have chosen static gauge — see section 2.

JHEP12(1999)022

field strength which now appears in the exponential factor. However, Douglas [13] has proposed that the background fields should be functionals of the nonabelian scalars (rather than, e.g., only the U(1) or center-of-mass component of Φ^i). Further it was pointed out [15] that the pull-backs of the bulk spacetime tensors should be defined in terms of covariant derivatives of the full nonabelian scalar fields. That is, $\partial_a\Phi^i \to D_a\Phi^i$ in eq. (1.2). Both of these suggestions were confirmed to leading order by examining string scattering amplitudes [16, 17]. Hence it would seem that extending the Chern-Simons action to accomodate the nonabelian world-volume theory involves simply introducing a gauge trace which encompasses all of the fields (and the pull-back) appearing in eq. (1.3). However, in the following section, we will argue that the resulting action is incomplete as it is not consistent with T-duality. Extending the action to one consistent with T-duality will involve introducing extra terms involving commutators of the nonabelian scalars. This construction reveals that Dp-branes also have natural couplings to the RR potentials with form degree larger than $p+1$.

We also use T-duality to build a consistent nonabelian extension of the Born-Infeld action (1.1). Our starting point is the D9-brane action which contains no scalars, and we simply apply a T-duality transformation on $9-p$ directions to produce that for a Dp-brane. The latter also generates new terms involving commutators of the nonabelian scalars. In incorporating the U(N) gauge symmetry here, we also include a single gauge trace[2] encompassing not only the explicit gauge fields and commutators, but also on the implicit nonabelian scalars in the functional dependence and pull-backs of the background fields [13, 15]. The appearance of the latter can again be confirmed for the Born-Infeld part of the action by examining string scattering amplitudes [16, 17].

An outline of the paper is as follows: We begin in section 2 by establishing our conventions and also recalling the T-duality transformations of both the spacetime and world-volume fields. We argue here that introducing a single gauge trace and substituting nonabelian fields everywhere in eqs. (1.1) and (1.3) yields an incomplete action. In section 3, we apply T-duality to construct a consistent nonabelian extension of the Born-Infeld action. In section 4, we present a similar construction for the Chern-Simons part of the action. In section 5, we discuss some consistency checks for the resulting actions. In particular, we compare the linear interactions of closed string fields to the nonabelian D0-brane action with the interactions derived for this case from matrix theory [20]. In section 6, we begin an investigation of the dynamics resulting from our action for Dp-branes moving in a nontrivial background fields, and discuss how the Dp-branes can be "polarized" by external fields. In a simple example, we show that a system of D0-branes in an external RR four-form field expands into a noncommutative two-sphere, which is interpreted as the formation of a

[2]Of course, there is some ambiguity as to the precise ordering of quantities inside this trace [18, 19], and we are only partially able to resolve this question below.

spherical D2-D0 bound state. We also compare these results with those calculated in the dual picture of a D2-brane carrying a nonvanishing U(1) field strength. The final section provides some discussion of our results.

2. Preliminaries

As we are interested in the dynamics of D-branes in nontrivial background fields, we begin by establishing our conventions for the massless bosonic fields in the type-II superstring theories. The bosonic part of the low-energy action for type-IIA string theory in ten dimensions may be written as (see e.g., [21])[3]

$$I_{IIA} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}\left\{e^{-2\phi}\left[R+4(\nabla\phi)^2-\frac{1}{12}H^2\right]-\frac{1}{4}(F^{(2)})^2-\frac{1}{48}(F^{(4)})^2\right\}- \\ -\frac{1}{4\kappa^2}\int BdC^{(3)}dC^{(3)}\,, \tag{2.1}$$

where $G_{\mu\nu}$ is the string-frame metric, ϕ is the dilaton, $H=dB$ is the field strength of the Neveu-Schwarz two-form, while the Ramond-Ramond field strengths are $F^{(2)}=dC^{(1)}$ and $F^{(4)}=dC^{(3)}+H\,C^{(1)}$.

For the type-IIB theory, we write the low-energy action as

$$I_{IIB} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}\left\{e^{-2\phi}\left[R+4(\nabla\phi)^2-\frac{1}{12}H^2\right]-\frac{1}{12}(F^{(3)}+C^{(0)}H)^2- \\ -\frac{1}{2}(\partial C^{(0)})^2-\frac{1}{480}(F^{(5)})^2\right\}+\frac{1}{4\kappa^2}\int\left(C^{(4)}+\frac{1}{2}BC^{(2)}\right)F^{(3)}H, \tag{2.2}$$

where the notation for the Neveu-Schwarz fields is the same as above, while $F^{(3)}=dC^{(2)}$ and $F^{(5)}=dC^{(4)}+H\,C^{(2)}$ are RR field strengths, and $C^{(0)}$ is the RR scalar. We are adopting the convention here that the the self duality constraint $F^{(5)}={}^*F^{(5)}$ is imposed by hand at the level of the equations of motion[24].

Given these actions, one may construct low energy background field solutions corresponding to various D-brane configurations — see, for example, refs. [25]. Note, however, that our conventions for the RR fields may differ slightly from the common choices in the supergravity literature.[4] The present choice coincides with the RR potentials appearing in the Chern-Simons action (1.3). This may be confirmed by verifying that the gauge invariances of the low-energy action are identical with those

[3]Further, the RR 9-form potential may be introduced to extend this (massless) IIA supergravity to the massive Romans supergravity [22] — see also [23].

[4]For example, translating to the fields in [26] requires:

$$(C^{(0)},C^{(1)},C^{(2)},C^{(3)},C^{(4)})=\left(\chi=A^{(0)},-A^{(1)},A^{(2)},A^{(3)},A^{(4)}-\frac{1}{2}B\,A^{(2)}\right).$$

of world-volume D-brane action. Further in the action, there is an overall factor of $(2\kappa^2)^{-1}$ which in particular multiplies the terms involving the RR fields. While this convention is standard in the literature on solutions of the supergravity equations (or some discussions of S-duality, e.g., refs. [27, 28] or matrix theory, e.g., ref. [20]), it is an unusual normalization of the RR fields compared to the standard discussions of D-brane physics — compare to ref. [3]. With regards to this point, our convention will be that the dilaton ϕ vanishes asymptotically, and then the ten-dimensional Newton's constant is given by $2\kappa^2 = 16\pi G_N = (2\pi)^7 \ell_s^8 g^2$ where g is the (asymptotic) closed string coupling. Finally, these actions are presented in terms of the string-frame metric, and as usual converting to the Einstein-frame metric is accomplished with

$$g_{\mu\nu} = e^{-\phi/2}\, G_{\mu\nu}\,. \tag{2.3}$$

Now we wish to recall the T-duality transformations of these supergravity fields. T-duality acts on the Neveu-Schwarz fields as [29]:

$$\begin{aligned}
\tilde{G}_{yy} &= \frac{1}{G_{yy}} & e^{2\tilde{\phi}} &= \frac{e^{2\phi}}{G_{yy}} \\
\tilde{G}_{\mu\nu} &= G_{\mu\nu} - \frac{G_{\mu y}G_{\nu y} - B_{\mu y}B_{\nu y}}{G_{yy}} & \tilde{G}_{\mu y} &= \frac{B_{\mu y}}{G_{yy}} \\
\widetilde{B}_{\mu\nu} &= B_{\mu\nu} - \frac{B_{\mu y}G_{\nu y} - G_{\mu y}B_{\nu y}}{G_{yy}} & \widetilde{B}_{\mu y} &= \frac{G_{\mu y}}{G_{yy}}
\end{aligned} \tag{2.4}$$

Here y denotes the Killing coordinate with respect to which the T-dualization is applied, while μ, ν denote any coordinate directions other than y. If y is identified on a circle of radius R, i.e. $y \sim y + 2\pi R$, then after T-duality the radius becomes $\tilde{R} = \alpha'/R = \ell_s^2/R$. The string coupling is also shifted as $\tilde{g} = g\ell_s/R$.

In section 3, it will be useful to implement T-duality on several Killing directions in a single transformation. Such a transformation is most easily implemented by first defining the ten-by-ten matrix [30]

$$E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}\,. \tag{2.5}$$

Then acting with T-duality on a set of directions denoted with $i, j = p+1, \ldots, 9$, the fields transform as:

$$\tilde{E}_{ab} = E_{ab} - E_{ai}E^{ij}E_{jb}\,, \qquad \tilde{E}_{aj} = E_{ak}E^{kj}\,, \qquad \tilde{E}_{ij} = E^{ij}\,, \tag{2.6}$$

where $a, b = 0, 1, \ldots, p$ denote the remaining coordinate directions. Here E^{ij} denotes the inverse of E_{ij}, i.e. $E^{ik}E_{kj} = \delta^i{}_j$. One also has the dilaton transformation

$$e^{2\tilde{\phi}} = e^{2\phi}\, \det(E^{ij})\,. \tag{2.7}$$

Of course, similar statements as above apply with regards to transforming the periodicities of the x^i directions and the string coupling constant g.

JHEP12(1999)022

Recall that T-duality transforms the type-IIA theory into the type-IIB theory and vice versa, through its action on the world-sheet spinors [31, 5]. This aspect of T-duality is then apparent in the transformations of the RR fields. The odd-form potentials of the IIA theory are traded for even-form potentials in the IIB theory and vice versa. Using the conventions adopted above, the transformation rules for the RR potentials are [32]:

$$\begin{aligned}\tilde{C}^{(n)}_{\mu\cdots\nu\alpha y} &= C^{(n-1)}_{\mu\cdots\nu\alpha} - (n-1)\frac{C^{(n-1)}_{[\mu\cdots\nu|y}G_{|\alpha]y}}{G_{yy}}\,,\\ \tilde{C}^{(n)}_{\mu\cdots\nu\alpha\beta} &= C^{(n+1)}_{\mu\cdots\nu\alpha\beta y} + nC^{(n-1)}_{[\mu\cdots\nu\alpha}B_{\beta]y} + n(n-1)\frac{C^{(n-1)}_{[\mu\cdots\nu|y}B_{|\alpha|y}G_{|\beta]y}}{G_{yy}}\,,\end{aligned} \tag{2.8}$$

where in general one would have $C^{(n)}$ with $n = 0, 2, 4, 6, 8, 10$ for the type-IIB theory, and $n = 1, 3, 5, 7, 9$ for type-IIA theory — $n = 9$ and 10 being exceptional cases [2]. This range of n is consistent with the implicit summation over RR potentials in the world-volume action (1.3). However, recall that these are not all independent, rather they appear in dual pairs (for $n \leq 8$) where at linear order $*dC^n = (-)^{n(n-1)/2}dC^{(8-n)}$. Of course, this is why the low-energy supergravity theories (2.1) and (2.3) are written in terms of RR potentials with only $n \leq 4$. The T-duality transformations (2.8) are consistent with those derived from the supergravity actions [21].

We now turn to consider the Dp-branes. To begin, we remark that throughout we are employing static gauge. That is first we employ spacetime diffeomorphisms to define the fiducial world-volume as $x^i = 0$ with $i = p+1, \ldots, 9$, and then with world-volume diffeomorphisms, we match the internal coordinates with the remaining spacetime coordinates on that surface, $\sigma^a = x^a$ with $a = 0, 1, \ldots, p$. With this choice and the identification $x^i = 2\pi\ell_s^2\Phi^i$, the general formula for pull-backs, e.g.,

$$P[G_{ab}] = G_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^a}\frac{\partial x^\nu}{\partial\sigma^b} \tag{2.9}$$

reduces to that given in eq. (1.2).

As described in the introduction, the low-energy dynamics of N coincident D-branes is described by a nonabelian U(N) gauge theory [11]. Our conventions are such that

$$A_a = A_a^{(n)}T_n, \qquad F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b] \tag{2.10}$$

where T_n are N^2 hermitean generators[5] with $\mathrm{Tr}(T_n\,T_m) = N\,\delta_{nm}$. The gauge fields are accompanied by $9-p$ adjoint scalars Φ^i with

$$D_a\Phi^i = \partial_a\Phi^i + i[A_a, \Phi^i]\,. \tag{2.11}$$

[5]There are certain subtleties to describing D-branes on a compact space [33, 34], but we will not have to consider these in the following.

Note that our conventions are such that both the gauge fields and adjoint scalars have the dimensions of $(\text{length})^{-1}$ — hence the appearance of the string scale in $x^i = 2\pi\ell_s^2\Phi^i$.

Now T-duality acts to change the dimension of a Dp-brane's world-volume [3]. The two possibilities are:

(i) if a coordinate transverse to the Dp-brane, e.g., $y = x^{p+1}$, is T-dualized, it becomes a D$(p+1)$-brane where y is now the extra world-volume direction;

(ii) if a world-volume coordinate on the Dp-brane, e.g., $y = x^p$, is T-dualized, it becomes a D$(p-1)$-brane where y is now an extra transverse direction.

In the process, the role of the corresponding world-volume fields change as

$$\text{(i)}\ \Phi^{p+1} \to A_{p+1}\,, \qquad \text{(ii)}\ A_p \to \Phi^p\,, \tag{2.12}$$

while the remaining components of A and scalars Φ are left unchanged.

If we focus on case (ii), the corresponding field strengths appearing in the world-volume action, (1.1) and (1.3), transform accordingly

$$F_{ap} \longrightarrow D_a\Phi^p\,. \tag{2.13}$$

We will show below that the modified interactions appear precisely in the correct way to yield the pull-backs (1.2) to the reduced world-volume. The pull-backs to the original Dp-brane world-volume will in general involve $D_p\Phi^i$. Now one would apply a T-duality transformation along x^p only when all fields are independent of this coordinate. Hence if we were considering a single Dp-brane with abelian world-volume fields, we would have

$$D_p\Phi^i = \partial_p\Phi^i = 0 \tag{2.14}$$

and so T-duality would not generate any new interactions from the pull-backs. However, in the nonabelian case, we would only require

$$D_p\Phi^i = \partial_p\Phi^i + i[A_p, \Phi^i] = i[A_p, \Phi^i]\,. \tag{2.15}$$

which need not vanish in general.[6] Hence T-duality would yield

$$D_p\Phi^i \longrightarrow i[\Phi^p, \Phi^i]\,. \tag{2.17}$$

[6]Here, one can think that the scalars Φ^i are constant along x^p up to a gauge transformation

$$\mathcal{L}_v\Phi^i = v^a D_a\Phi^i = i[\Lambda, \Phi^i]\,, \tag{2.16}$$

where $v^a\partial_a = \partial_p$ is the Killing vector in the x^p-direction, and the gauge parameter would be $\Lambda = v^a A_a$. This point of view is analogous to the generalized analysis of T-duality for spacetime fields in ref. [37].

generating new world-volume interactions involving scalar commutator terms. These interactions would not be included if we simply introduced an overall gauge trace and substituted nonabelian fields everywhere in eqs. (1.1) and (1.3), and hence the resulting actions are inconsistent with T-duality. It is taking care to retain all of the commutators, which distinguishes the present investigation from previous discussions of T-duality in the context of D-branes [35, 36].

With the present conventions, the tension of an individual Dp-brane, i.e. the coefficient appearing in the Born-Infeld term (1.1), is as usual [3]

$$T_p = \frac{2\pi}{g(2\pi\ell_s)^{p+1}}\,. \tag{2.18}$$

However, because of the "unconventional" normalization of the RR fields noted above, the charge of an individual Dp-brane, i.e. the coefficient appearing in the Chern-Simons action (1.3), is

$$\mu_p = \frac{2\pi}{g(2\pi\ell_s)^{p+1}}\,. \tag{2.19}$$

Hence we have $\mu_p = T_p$, which simplifies some of our calculations. Finally, in the following discussion, it will be convenient to define:

$$\lambda = 2\pi\ell_s^2\,. \tag{2.20}$$

3. Born-Infeld action

In the previous section, we showed that merely extending the abelian Born-Infeld action (1.1) with the substitution of nonabelian fields and an overall gauge trace would miss certain interactions involving commutators of the scalar fields. For example, in a particular limit, the correct action for N coincident Dp-branes should reduce to $(p+1)$-dimensional U(N) super Yang-Mills theory which has a scalar potential proportional to $\mathrm{Tr}\,[\Phi^i,\Phi^j]^2$. This potential would not appear in the naive construction suggested above.

Our approach will be to apply this construction, but only for the D9-brane action to provide a starting point. In this case, there are no world-volume scalar fields, because the D-brane fills the entire space and there are no transverse directions. Given the absence of any scalars, making the naive construction will not omit the commutator terms described above.[7] Thus our nonabelian Born-Infeld action for the D9-brane is

$$S_{BI} = -T_9 \int d^{10}\sigma\ \mathrm{Tr}\left(e^{-\phi}\sqrt{-\det\left(G_{ab}+B_{ab}+\lambda\,F_{ab}\right)}\right)\,. \tag{3.1}$$

[7] Of course, we could have also begun with the D8-brane where again there can be no nontrivial commutators because there is a single transverse scalar field.

Note that in the absence of any transverse directions, there is no need to introduce a pull-back on $G_{ab} + B_{ab}$. Now our strategy is to apply a T-duality transformation, using eqs. (2.6), (2.7) and (2.12), on $9-p$ coordinates x^i with $i = p+1, \ldots, 9$ to eq. (3.1) to produce the corresponding Born-Infeld action for a Dp-brane. Note that in eq. (3.1), the background fields are in general functions of all of the world-volume coordinates. For the purposes of the T-duality transformation, we must consider the special case that the fields are independent of the coordinates x^i. This produces a Dp-brane action functional of the background fields for this special case, but we assume the extension to the general case is simply to allow the background fields in the new action to be functions of all of the spacetime coordinates. The latter entails introducing a functional dependence on the nonabelian scalars — see eq. (5.1) below — which may seem a radical alteration, but recall that string scattering amplitudes have already provided evidence for this structure [16, 17].

Focusing on the determinant in the D9-brane action, in the notation of eq. (2.5), we have

$$D = \det\left(E_{ab} + \lambda\, F_{ab}\right) \tag{3.2}$$

and then applying the T-duality transformation rules (2.6) and (2.12) yields

$$\tilde{D} = \det\begin{pmatrix} E_{ab} - E_{ai}E^{ij}E_{jb} + \lambda\, F_{ab} & E_{ak}E^{kj} + \lambda\, D_a\Phi^j \\ -E^{ik}E_{kb} - \lambda\, D_b\Phi^i & E^{ij} + i\lambda\,[\Phi^i, \Phi^j] \end{pmatrix}, \tag{3.3}$$

where as before, E^{ij} denotes the inverse of E_{ij}.

Now it is instructive to consider this construction with the abelian theory for which the commutators vanish and $D_a\Phi^i = \partial_a\Phi^i$. In this case, the matrix inside the determinant above may be manipulated to give

$$\begin{aligned}
\tilde{D} &= \det\begin{pmatrix} \begin{matrix} E_{ab} - E_{ai}E^{ij}E_{jb} + \lambda\, F_{ab} \\ +(E_{ak}E^{kl} + \lambda\,\partial_a\Phi^l)(E_{lb} + \lambda\, E_{lm}\partial_b\Phi^m) \end{matrix} & E_{ak}E^{kj} + \lambda\,\partial_a\Phi^j \\ 0 & E^{ij} \end{pmatrix} \\
&= \det\begin{pmatrix} \begin{matrix} E_{ab} + \lambda\,\partial_a\Phi^k E_{kb} + \lambda\, E_{ak}\partial_b\Phi^k \\ +\lambda^2\,\partial_a\Phi^k E_{kl}\partial_b\Phi^l + \lambda\, F_{ab} \end{matrix} & E_{ak}E^{kj} + \lambda\,\partial_a\Phi^j \\ 0 & E^{ij} \end{pmatrix} \\
&= \det\left(P[E]_{ab} + \lambda\, F_{ab}\right)\, \det(E^{ij})\,.
\end{aligned} \tag{3.4}$$

Hence the scalar derivative terms which were generated by T-duality provide precisely necessary terms to yield the pull-back of E_{ab} to the new reduced world-volume. Next by the transformation rule (2.7), the dilaton factor in eq. (3.1) is replaced by

$$e^{-\phi} \to \frac{e^{-\phi}}{\sqrt{E^{ij}}}\,, \tag{3.5}$$

which provides precisely the necessary factor to cancel the second determinant above when the latter appears under the square root in the action. In the D9-brane action, the invariant directions, i.e. x^i, can be integrated out before the T-duality transformation so that the overall prefactor becomes $T_9\prod_{i=p+1}^{9}(2\pi R_i)$. Now taking into account the transformations of the radii, $R_i \to \ell_s^2/R_i$, and of the string coupling in T_9, $g \to g\ell_s^{9-p}/\prod_{i=p+1}^{9} R_i$, this leaves an overall factor of precisely T_p in front of the T-dual action [3]. Assembling all of these factors yields the T-dual action

$$\tilde{S}_{BI} = -T_p \int d^{p+1}\sigma \, \left(e^{-\phi}\sqrt{-\det(P[G+B]_{ab} + \lambda\, F_{ab})}\right) , \tag{3.6}$$

which has precisely the desired form. Thus we have recovered the result that in the abelian theory, the Born-Infeld action (1.1) is compatible with T-duality [35].

Unfortunately the full determinant (3.3) including the commutator terms does not lead to such a simple result as in eq. (3.4). In this case, it is useful to define

$$Q^i{}_j \equiv \delta^i{}_j + i\lambda\, [\Phi^i, \Phi^k]\, E_{kj}\, . \tag{3.7}$$

Then following manipulations as in eq. (3.4), one finds

$$\tilde{D} = \det\left(P\left[E_{ab} + E_{ai}(Q^{-1} - \delta)^{ij} E_{jb}\right] + \lambda\, F_{ab}\right)\, \det(E^{ij})\, \det(Q^i{}_j)\, . \tag{3.8}$$

Note the second index on the expression $(Q^{-1} - \delta)^{ij}$ has been raised using E^{ij} (rather than G^{ij}). The remaining ingredients of the calculation are precisely as above, and the final T-dual action becomes

$$\tilde{S}_{BI} = -T_p \int d^{p+1}\sigma \, \mathrm{Tr}\left(e^{-\phi}\sqrt{-\det\left(P\left[E_{ab} + E_{ai}(Q^{-1} - \delta)^{ij} E_{jb}\right] + \lambda F_{ab}\right) \det(Q^i{}_j)}\right) . \tag{3.9}$$

An interesting features of this result is that it now contains a product of two determinants.[8] It is the second determinant which supplies the standard scalar potential in the flat space limit. Setting $G_{\mu\nu} = \eta_{\mu\nu}$ and $B_{\mu\nu} = 0$, one may expand this factor to find

$$\sqrt{\det\, Q^i{}_j} = 1 - \frac{\lambda^2}{4}[\Phi^i, \Phi^j][\Phi^i, \Phi^j] + \cdots \tag{3.10}$$

Of course, this expansion also contains higher order potential terms, and in general, more commutator terms arise in the first determinant factor through the expression $(Q^{-1} - \delta)^{ij}$.

Now the Born-Infeld action is highly nonlinear, and so the proposed action (3.9) is incomplete without a precise prescription for how the gauge trace should be implemented. To incorporate the nonabelian symmetry of the world-volume fields Tseytlin [18] — see also [19] — has suggested that one should supplement the usual Born-Infeld form (for a flat space background) with a symmetrized trace over gauge indices. That is, to leading order all commutators of the field strengths should be dropped. Unfortunately this action does not seem to capture the full physics of the

[8] This feature was noted for the flat space action in ref. [19].

infrared limit [38], and it appears that in general nontrivial commutators must be included at sixth-order in the field strength [39, 40]. On the other hand in configurations with supersymmetry, this action seems to properly describe the physics of the nonabelian Yang-Mills fields [41], and even provides solutions of the full open-string equations of motion [42].

In the above action (3.9), we will adopt the same symmetrized trace prescription. That is, the trace is completely symmetric between all nonabelian expressions of the form F_{ab}, $D_a\Phi^i$ and $i[\Phi^i,\Phi^j]$. In part, this prescription is adopted as a practical matter as these objects were treated as commuting in all of the manipulations leading up to eq. (3.9). Given the previous results [39, 40] then, we expect that in general this action will only incorporate the correct interactions involving these objects at second and fourth order. So for example, potential terms arising from the expansion of $\det(Q)$ in eq. (3.10) may not be reliable for sixth and higher orders in the commutators. However, there is some hope that it will still properly describe the physics of supersymmetric configurations to all orders.

This discussion has not addressed how the gauge trace should be implemented for the nonabelian scalars appearing in the functional dependence of the background fields. We will leave this question until section 5.

4. Chern-Simons action

For the abelian theory of an individual D-brane, one can show that the Chern-Simons action (1.3) is compatible with T-duality [35]. In general the analysis is quite complicated because of the interplay of the RR and the Neveu-Schwarz fields.[9] For example, T-duality (2.4) acting on the two-form B in general introduces terms involving $G_{\mu y}$, but these are all precisely cancelled by the terms involving the metric appear in the transformation of RR potentials (2.8). The discussion, however, is greatly simplified for the case of flat space, i.e. $G_{\mu\nu}=\eta_{\mu\nu}$ and $B_{\mu\nu}=0$. In this case, the T-duality transformations of the RR potentials become [3]

$$\tilde{C}^{(n)}_{\mu\cdots\nu\alpha y}=C^{(n-1)}_{\mu\cdots\nu\alpha}\,,\qquad \tilde{C}^{(n)}_{\mu\cdots\nu\alpha\beta}=C^{(n+1)}_{\mu\cdots\nu\alpha\beta y}\,,\tag{4.1}$$

while the action (1.3) reduces to

$$S_{CS}=\mu_p\int P\left[\sum C^{(n)}\right]e^{\lambda F}\,.\tag{4.2}$$

Verifying the consistency of the form of this action with T-duality is relatively straightforward, and we sketch the calculation here. We consider the case where

[9]Recently there have been some interesting suggestions on the transformation properties of the RR fields for higher dimensional tori [43]. This formalism may simplify the analysis of the general case in the present discussion.

JHEP12(1999)022

T-duality acts on a world-volume coordinate, e.g., $y = x^p$, so the Dp-brane becomes a D$(p-1)$-brane. Now the integrand in eq. (4.2) is a form on the world-volume and so the index y will appear on precisely one tensor in each of the terms in the sum. For the abelian theory, the pull-back involves ordinary derivatives of the scalars, and so as $\partial_y \Phi^i = 0$, the existing pull-back terms are unaffected by the T-duality transformation, i.e. they already coincide with the pull-back to the new reduced world-volume from the transverse directions x^i with $i = p+1, \ldots, 9$. Hence we may consider a generic term of the form

$$P[C^{(n)}]_{a_1 \cdots a_n} F_{b_1 c_1} \cdots F_{b_m c_m} , \tag{4.3}$$

where again y appears as one and only one of the indices. If this index is carried by the RR potential, then the T-duality transformation (4.1) simply removes this index leaving

$$P'[C^{(n-1)}]_{a_1 \cdots a_{n-1}} F_{b_1 c_1} \cdots F_{b_m c_m} , \tag{4.4}$$

where the prime on the pull-back indicates that it only involves $\partial_a \Phi^i$ with $i = p+1, \ldots, 9$. The other possibility is that the y index is carried by one of the gauge field strengths, in which case eqs. (4.1) and (2.13) yield a contribution of the form

$$m\, P'[C^{(n+1)}]_{a_1 \cdots a_n y} F_{b_1 c_1} \cdots F_{b_{m-1} c_{m-1}} \partial_{b_m} \Phi^y . \tag{4.5}$$

Hence these contributions complete the pull-back P' which would have appeared in eq. (4.4) if we had begun with $n+2$ and $m-1$ in eq. (4.3). This gives the essence of the calculation. Checking the consistency in detail requires verification that a few factors and signs work out properly, and that the prefactor in the T-dual action becomes μ_{p-1} (in the same way as T_p is transformed in the previous section). In the end, one does find that the action (4.2) is compatible with T-duality [35]. The calculations become far more involved for the case of general background fields, i.e. arbitrary $G_{\mu\nu}$ and $B_{\mu\nu}$, however the consistency of the full Chern-Simons action (1.3) can still be verified on a case-by-case basis [35].

Now we argued in section 2 that the form action in eqs. (1.3) and (4.2) fails to include certain interactions involving the commutators of the nonabelian scalar fields. So let us simply present the nonabelian generalization of the flat space Chern-Simons action (4.2):

$$S_{CS} = \mu_p \int \mathrm{Tr}\left(P\left[e^{i\lambda\, \mathrm{i}_\Phi \mathrm{i}_\Phi} \sum C^{(n)} \right] e^{\lambda F} \right) . \tag{4.6}$$

Beyond the generalization of all of world-volume fields to their nonabelian counterparts (including in the functional dependence of the RR potentials) and the inclusion of an overall gauge trace, there is the appearance of the exponential $\exp(i\lambda\, \mathrm{i}_\Phi \mathrm{i}_\Phi)$ inside the pull-back. Here i_Φ denotes the interior product by Φ^i regarded as a vector in the transverse space. Acting on forms, the interior product is an anticommuting operator of form degree -1, e.g.,

$$\begin{aligned} C^{(2)} &= \frac{1}{2} C^{(2)}_{\mu\nu}\, dx^\mu dx^\nu \\ \mathrm{i}_v C^{(2)} &= v^\mu\, C^{(2)}_{\mu\nu}\, dx^\nu \\ \mathrm{i}_w \mathrm{i}_v C^{(2)} &= w^\nu v^\mu\, C^{(2)}_{\mu\nu} = -\mathrm{i}_v \mathrm{i}_w C^{(2)} \,. \end{aligned} \tag{4.7}$$

In particular, for ordinary vectors one has $(\mathrm{i}_v)^2 = 0$. The exponential makes a nontrivial contribution in eq. (4.6) because of the nonabelian nature of the displacement vectors Φ^i. There one has

$$\mathrm{i}_\Phi \mathrm{i}_\Phi C^{(2)} = \Phi^j \Phi^i\, C^{(2)}_{ij} = \frac{1}{2} C^{(2)}_{ij}\, [\Phi^j, \Phi^i] \tag{4.8}$$

and so the exponential produces precisely the desired commutators.

Verifying the consistency of the action (4.6) with T-duality follows essentially the same argument as above for the abelian action (4.2). Again, we outline the calculation where T-duality acts on a world-volume coordinate, $y = x^p$. The only real difference is that the pull-back now involves gauge covariant derivatives of the scalars. Hence since $D_y \Phi^i = i[A_y, \Phi^i]$ which in general is nonvanishing, T-duality acting on the pull-back will now give rise to new interactions. Here, we consider a generic term of the form

$$(i\lambda)^l P[(\mathrm{i}_\Phi \mathrm{i}_\Phi)^l C^{(n+2l)}]_{a_1 \cdots a_n}\, F_{b_1 c_1} \cdots F_{b_m c_m} \,, \tag{4.9}$$

where again y appears as one and only one of the indices. First since the scalars above only carry indices $i = p+1, \ldots, 9$, the operator $(\mathrm{i}_\Phi \mathrm{i}_\Phi)^l$ will remain unaffected by the T-duality. If the y index is carried by the RR potential after the pull-back, then the T-duality transformation (4.1) simply removes this index leaving

$$\frac{(i\lambda)^l}{l!} P'[(\mathrm{i}_{\hat\Phi} \mathrm{i}_{\hat\Phi})^l C^{(n+2l-1)}]_{a_1 \cdots a_{n-1}}\, F_{b_1 c_1} \cdots F_{b_m c_m} \,, \tag{4.10}$$

where the prime on the pull-back indicates that it only involves $D_a \Phi^i$ with $i = p+1, \ldots, 9$, and similarly $\mathrm{i}_{\hat\Phi}$ indicates that the displacement vector in the interior product only has components in these same directions, i.e. the original transverse space. Another possibility is that y appears as one of the indices a_k in eq. (4.9), but that it is actually carried by $D_y \Phi^i$ in the pull-back. In this case, the T-duality transformation yields

$$2\frac{(i\lambda)^{l+1}}{l!}\, P'[(\mathrm{i}_{\hat\Phi} \mathrm{i}_{\hat\Phi})^l (\mathrm{i}_{\Phi^y} \mathrm{i}_{\hat\Phi}) C^{(n+2l+1)}]_{a_1 \cdots a_{n-1}}\, F_{b_1 c_1} \cdots F_{b_m c_m} \,. \tag{4.11}$$

The role of these terms is to complete the interior product operators that would appear in a term given in eq. (4.10) if one replaced l by $l+1$ in eq. (4.9). That is, one can write

$$\frac{(i\lambda)^{l+1}}{(l+1)!} (\mathrm{i}_\Phi \mathrm{i}_\Phi)^{l+1} = \frac{(i\lambda)^{l+1}}{(l+1)!} (\mathrm{i}_{\hat\Phi} \mathrm{i}_{\hat\Phi})^{l+1} + 2\frac{(i\lambda)^{l+1}}{l!} (\mathrm{i}_{\hat\Phi} \mathrm{i}_{\hat\Phi})^l (\mathrm{i}_{\Phi^y} \mathrm{i}_{\hat\Phi}) \tag{4.12}$$

when one realizes that Φ^y can at most appear in one of the interior products since the combined operator is acting on a form. The final possibility, as in the abelian case, is that the y index is carried by one of the gauge field strengths, in which case eqs. (4.1) and (2.13) yield a contribution of the form

$$m\,(i\lambda)^l P'[(\mathrm{i}_\Phi \mathrm{i}_\Phi)^l C^{(n+2l+1)}]_{a_1\cdots a_n y}\, F_{b_1 c_1}\cdots F_{b_{m-1}c_{m-1}}\, D_{b_m}\Phi^y\,. \tag{4.13}$$

As before the role of these contributions is to complete the pull-back P' which would have appeared in eq. (4.10) if we had begun with $n+2$ and $m-1$ in eq. (4.9). One point to note is that in collecting contributions to form interactions of the form given in eq. (4.9) but now on the reduced world-volume with $a=0,\ldots,p-1$, one will never find i_{Φ^y} and $D_a\Phi^y$ in the same term as they cannot both be contracted on the same RR form. The above discussion describes the core of the consistency calculations. Again we have left out some of the details, and simply state that one will find T-duality will transform the action (4.6) for a Dp-brane to the D$(p-1)$-brane action with the same precise form. The reader is invited to verify these details for herself.

In extending eq. (4.6) beyond the case of flat space, the Chern-Simons action for the nonabelian world-volume theory becomes

$$S_{CS} = \mu_p \int \mathrm{Tr}\left(P\left[e^{i\lambda\,\mathrm{i}_\Phi \mathrm{i}_\Phi}(\sum C^{(n)}\,e^B)\right]e^{\lambda F}\right)\,. \tag{4.14}$$

It is clear that this action reduces to the expected form (1.3) for the abelian theory of a single Dp-brane. The previous discussion verifying consistency with T-duality of the action (4.6) for flat space generalizes here in a straightforward way for more general backgrounds, as long as we impose the restriction that $G_{\mu y}=0$ which keeps the transformations (2.8) of the RR potentials relatively simple. One fact that is revealed by such an analysis is that the interior products in eq. (4.14) must act on both the Neveu-Schwarz two-form and the RR potentials. For example, one would find such interior products in eq. (4.11) if the general transformation of the RR potential was applied in the previous discussion. Calculations to verify that eq. (4.14) is consistent with T-duality $G_{\mu\nu}$ is completely general become much more involved, however, we have verified the consistency of this action (4.14) in certain specific cases, e.g., considering the D2-brane with arbitrary $G_{\mu\nu}$ and $B_{\mu\nu}$, but $C^{(n)}=0$ for $n\geq 5$.

Let us make a few remarks on the interpretation of eq. (4.14). First, note that the integrand is to be evaluated by considering the expression in each set of brackets in turn, from the innermost around $\sum C^{(n)}e^B$ to the outermost for the gauge trace. In particular, this means that first $\sum C^{(n)}e^B$ is expanded as a sum of forms in the ten-dimensional spacetime, and so only a finite set of terms in the exponential will contribute. Then the exponential $\exp(i\lambda\,\mathrm{i}_\Phi\mathrm{i}_\Phi)$ acts on this sum of forms, and so again only a finite set of terms in this second exponential will contribute.

Second, just as in the previous section with the Born-Infeld action, the proposed action (4.14) is incomplete without a precise prescription for how the gauge trace should be implemented. Here, we will adopt the same symmetrized trace prescription as in the Born-Infeld action. That is, the trace is completely symmetric between all nonabelian expressions of the form F_{ab}, $D_a\Phi^i$ and $i[\Phi^i,\Phi^j]$. Further here we must extend this symmetrization to the individual background field components, each of which is in general a functional of the nonabelian scalars. This symmetrization must be adopted here as a practical matter as all of these objects were treated as commuting in the calculations which verified that eq. (4.14) was compatible with T-duality. Given the complicated possible functional dependence on Φ^i which could appear in the background fields, the above description of the gauge trace is still not complete. We will leave the question of its precise implementation to the following section.

It is well-known that a Dp-brane couples not only to the RR potential with form degree $n = p+1$ [2], but also that it can couple to the RR potentials with $n = p-1, p-3, \ldots$ through the additional interactions with the two-form B and world-volume field strength F appearing in the action (1.3) [9]. Here we see from the nonabelian action (4.14) that a Dp-brane can also couple to the RR potentials with $n = p+3, p+5, \ldots$ through the additional interactions involving commutators of the nonabelian scalars. To make these couplings more explicit, consider the D0-brane action (for which F vanishes):

$$\begin{aligned} S_{CS} = \mu_0 \int \mathrm{Tr}\bigg(P\bigg[& C^{(1)} + i\lambda\, \mathrm{i}_\Phi \mathrm{i}_\Phi \left(C^{(3)} + C^{(1)}B\right) - \\ & - \frac{\lambda^2}{2}(\mathrm{i}_\Phi\mathrm{i}_\Phi)^2 \left(C^{(5)} + C^{(3)}B + \frac{1}{2}C^{(1)}B^2\right) - \\ & - i\frac{\lambda^3}{6}(\mathrm{i}_\Phi\mathrm{i}_\Phi)^3 \left(C^{(7)} + C^{(5)}B + \frac{1}{2}C^{(3)}B^2 + \frac{1}{6}C^{(1)}B^3\right) + \\ & + \frac{\lambda^4}{24}(\mathrm{i}_\Phi\mathrm{i}_\Phi)^4 \left(C^{(9)} + C^{(7)}B + \frac{1}{2}C^{(5)}B^2 + \frac{1}{6}C^{(3)}B^3 + \frac{1}{24}C^{(1)}B^4\right)\bigg]\bigg) \\ = \mu_0 \int dt\, \mathrm{Tr}\bigg(& C^{(1)}_t + \lambda C^{(1)}_i D_t\Phi^i + i\frac{\lambda}{2}(C^{(3)}_{tjk}[\Phi^k,\Phi^j] + \lambda C^{(3)}_{ijk} D_t\Phi^i[\Phi^k,\Phi^j]) + \cdots \bigg), \end{aligned} \tag{4.15}$$

where we assume that $\sigma^0 = t$ in static gauge. We see here that the nonabelian action (4.14) is giving interactions reminiscent of those appearing in matrix theory [44, 45]. This similarity is, of course, no accident, as we will see in the next section by comparing to the couplings derived from matrix theory by Taylor and van Raamsdonk [20]. For example, there is linear coupling to C^3, which is the potential corresponding to D2-brane charge,

$$i\lambda\,\mu_0 \int \mathrm{Tr}\, P\left[\mathrm{i}_\Phi\mathrm{i}_\Phi C^{(3)}\right] = i\frac{\lambda}{2}\mu_0 \int dt\ \mathrm{Tr}\left(C^{(3)}_{tjk}\,[\Phi^k,\Phi^j] + \lambda C^{(3)}_{ijk}\, D_t\Phi^k\,[\Phi^k,\Phi^j]\right). \tag{4.16}$$

Note that the first term on the right hand side has the form of a source for D2-brane charge. This is essentially the interaction central to the construction of D2-branes in matrix theory with the large-N limit [44, 45]. Here, however, with finite N, this term vanishes upon taking the trace if $C^{(3)}_{tjk}$ was simply a constant or a function of the world-volume coordinate t. However, in general one should regard these components of the RR three-form as functionals of Φ^i. Hence, while there would be no "monopole" coupling to D2-brane charge, nontrivial expectation values of the scalars can give rise to couplings to an infinite series of higher "multipole" moments. In addition, we will discuss in section 6 that in a nontrivial background $C^{(3)}$, this interaction gives rise to additional terms in the potential for the scalars.

5. Comparison with matrix theory

Thus far, we have constructed a nonabelian world-volume action for test Dp-branes in general background fields. The action consists of two parts, the Born-Infeld term (3.9) and the Chern-Simons term (4.14). Our construction was guided by the simple principle that the result should be consistent with the familiar rules of T-duality. In this section, we would like to discuss how our results measure up to some previous considerations of nonabelian Dp-brane actions. This exercise will also be useful in providing a precise prescription for the implementation of the gauge trace in our action — see the comparison to matrix theory, below.

First, we will remark on the connection of our Born-Infeld action (3.9) to Tseytlin's proposal [18] for a nonabelian Born-Infeld action for ten-dimensional super Yang-Mills theory. Tseytlin's action is, of course, precisely recovered in eq. (3.9) with $p = 9$, i.e. eq. (3.1) and a flat space background, i.e. $G_{\mu\nu} = \eta_{\mu\nu}$ and $B_{\mu\nu} = 0$. This action [18] correctly yields the F^2 and F^4 interactions expected for low energy superstring theory [46], however, it seems to require modifications [39, 40] at order F^6. Introducing a general metric (but keeping $B_{\mu\nu} = 0$) in our D9-brane action (3.1) simply covariantizes the super Yang-Mills action with minimal coupling to the metric, i.e. the same F^2 and F^4 (and all higher order) interactions are recovered but now the index contractions are made with the curved space metric. Of course, T-duality requires the inclusion of B in eq. (3.1), since T-duality (2.4) interchanges components of the metric and this two-form. As usual, the factor of $\exp(-\phi)$ is required in the action as it must reproduce interactions derived from superstring disk amplitudes [3]. Hence we are arguing that eq. (3.1), our starting point in the construction of the Born-Infeld action (3.9), provides the minimal extension of Tseytlin's action [18] to include nontrivial Neveu-Schwarz backgrounds. As mentioned above, the connection with Tseytlin's action also points out the limitations of eq. (3.9), namely that we must expect that there are corrections at order six in F_{ab}, $D_a\Phi^i$ and/or $i[\Phi^i, \Phi^j]$. Given that the precise form of the sixth order corrections to Tseytlin's action are known [39], presumably those to eq. (3.9) could be determined by covariantizing the flat space corrections, multiplying by $\exp(-\phi)$, and demanding consistency with T-duality.

JHEP12(1999)022

Douglas [13] observed that whatever their form, the nonabelian world-volume actions should contain a single gauge trace, as do both eqs. (3.9) and (4.14). This observation again stems from the fact that these actions are encoding the low energy interactions derived from disk amplitudes in superstring theory. Since the disk has a single boundary, the single gauge trace arises from the standard open string prescription of tracing over Chan-Paton factors on each world-sheet boundary. In particular, this means that the background (closed string) fields appearing in the action cannot be functionals of the D-brane center-of-mass coordinate $x^i = \frac{\lambda}{N}\,\mathrm{Tr}\,\Phi^i$. Hence, it is natural to assume that the background fields are functionals of the nonabelian scalar fields instead [13]. One can sharpen this reasoning by observing that the only difference in the superstring amplitudes between the U(1) and the U(N) theories is that the amplitudes in the latter case are multiplied by an additional trace of Chan-Paton factors. Hence up to commutator "corrections," the low energy interactions should be the same in both cases. Hence since the background fields are functionals of the neutral U(1) scalars in the abelian theory, they must be precisely the same functionals of the adjoint scalars in the nonabelian theory, up to commutator corrections. Similarly the pull-back was constructed with neutral scalars in the abelian theory, and so the adjoint scalars play the same role in the nonabelian theory [15].

While we have indicated that the background fields are functionals of the nonabelian scalars, let us be more precise on how this is actually implemented [16]. For example, consider the case of a Dp-brane propagating in a curved background $G^0_{\mu\nu}(x^\rho)$. First we impose static gauge as described above eq. (2.9), and hence the spacetime coordinates are split into world-volume coordinates, $x^a = \sigma^a$, and transverse coordinates, x^i. Then the metric functional appearing in the D-brane action would be given by a nonabelian Taylor expansion [16]

$$\begin{aligned} G_{\mu\nu} &= \exp\left[\lambda\Phi^i\,\partial_{x^i}\right]\,G^0_{\mu\nu}(\sigma^a,x^i)|_{x^i=0} \\ &= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\,\Phi^{i_1}\cdots\Phi^{i_n}\,(\partial_{x^{i_1}}\cdots\partial_{x^{i_n}})G^0_{\mu\nu}(\sigma^a,x^i)|_{x^i=0}\,. \end{aligned} \tag{5.1}$$

The same nonabelian Taylor expansions arise in the context of matrix theory, e.g. [47, 48]. Since the partial derivatives above all commute, there does not seem to be room for commutator corrections to this Taylor expansion, i.e. it naturally produces an expression symmetric in all of the Φ^{i_m}. The ambiguities come in the full nonlinear action where the trace must be implemented over Φ^i's from different background field components, as well nonabelian terms such as F_{ab}.

We also mention that explicit calculations of superstring amplitudes [16, 17, 49] yield a detailed agreement with the actions given in eqs. (3.9) and (4.14). In particular, disk amplitudes with one closed string state and two open strings [16] reveal terms to second order in the nonabelian Taylor expansion (5.1), as well as contributions of the adjoint scalars to the pull-backs. Some disk amplitudes with one closed and three open strings have been evaluated [17], and here some of the single commutator corrections can be seen. For example, one sees the commutator arising from the interior products in eq. (4.16).

The main comparison which we wish make is between certain interactions in eqs. (3.9) and (4.14), and those derived from matrix theory [20]. Taylor and van Raamsdonk [20] have recently calculated all of the linear couplings of (massless bosonic) closed string fields to D0-branes in type-IIA string theory, by transcribing the analogous couplings to D=11 supergravity fields derived for matrix theory [47, 48] in the Sen-Seiberg limit [50]. The nonabelian action proposed here contains analogous linear couplings (as well as nonlinear couplings). A detailed comparison the linear couplings derived by these the two approaches yields an precise agreement, to the order that the calculations are expected to be valid, up to two caveats, which will be discussed below.[10] I will illustrate these comparisons by providing the detailed expressions for two cases, the linear couplings of the dilaton and the RR five-form potential.

Taylor and van Raamsdonk [20] present the linear coupling to the dilaton as

$$\int ds \sum_{n=0}^{\infty} \frac{1}{n!} \left(\partial_{x^{i_1}} \cdots \partial_{x^{i_n}} \phi\right) I_\phi^{(i_1 \cdots i_n)} \,, \tag{5.2}$$

where $\partial_{x^{i_1}} \cdots \partial_{x^{i_n}} \phi(s)$ are the derivatives of the background dilaton field evaluated along the D0 world-line, i.e. the c-number coefficients in the Taylor series for the background field (5.1). The various moments have the form

$$I_\phi^{(i_1 \cdots i_n)} = \mathrm{STr}(I_\phi, X^{i_1}, \ldots, X^{i_n}) \,, \tag{5.3}$$

where the X^i are the nonabelian matrix displacements completing the Taylor expansion (5.1) of the background field. In eq. (5.3), we have dropped an additional term including extra fermionic contributions to the higher multipole moments — as we have not included the world-volume fermions in any of our calculations, we can make no comparison for these terms. As well as the nonabelian displacements, these currents are constructed with I_ϕ, the zeroth moment of the background field coupling. The latter is an expression constructed from $D_s X^i$ and $i[X^i, X^j]$, as well as Θ and $[X^i, \Theta]$ — these fermionic terms will again be dropped for the purposes of the present comparisons. The symmetrized trace indicated by $\mathrm{STr}(\cdots)$ indicates that the gauge trace takes a symmetrized average over all orderings of factors of $D_s X^i$ and $i[X^i, X^j]$, and the individual nonabelian displacements X^i, appearing in the higher moments. This form of the coupling generalizes to all of the background fields, and so the essential step is now to identify the form of the zeroth moment currents.

[10] A further caveat is that certain ambiguous signs in the matrix model result are actually fixed by comparing to the D-brane couplings [51].

From the expressions provided by Taylor and van Raamsdonk [20], one determines the zeroth moment of the dilaton coupling to be

$$
\begin{aligned}
I_\phi = \frac{1}{R}\Bigg(&1 - \frac{1}{2}D_sX^iD_sX^i - \frac{1}{4}[X^i,X^j][X^i,X^j] - \frac{1}{8}(D_sX^iD_sX^i)^2 - \\
&- \frac{1}{8}[X^i,X^j][X^j,X^k][X^k,X^l][X^l,X^i] + \frac{1}{32}([X^i,X^j][X^j,X^i])^2 - \\
&- \frac{1}{8}[X^i,X^j][X^j,X^i]D_sX^kD_sX^k + \frac{1}{2}D_sX^i[X^i,X^j][X^j,X^k]D_sX^k\Bigg) + \\
&+ O_c(v^4) + O(v^6)\,, \qquad (5.4)
\end{aligned}
$$

where R is the compactification radius of the tenth spatial dimension in M-theory. This expression gives the leading order contributions in an expansion in small D_sX^i and $i[X^i, X^j]$. The $O(v^6)$ indicates that there will be new contributions at sixth order in these expressions, while the $O_c(v^4)$ indicates that there could also be additional commutator corrections at fourth order in the expansion. Here we have again dropped all of the fermion contributions calculated in ref. [20].

Now let us compare to the linear dilaton coupling that arises in D-brane action derived here. Expanding the Born-Infeld term (3.9) around flat space keeping only the term linear in the dilaton yields

$$
T_0 \int dt\ \mathrm{Tr}\left[\phi(t,\Phi)\sqrt{(1-\lambda^2 D_t\Phi^i\,Q^{-1}_{ij}\,D_t\Phi^j)\ \det Q_{ij}}\right], \qquad (5.5)
$$

where from eq. (3.7),

$$
Q_{ij} = \delta_{ij} + i\lambda[\Phi^i,\Phi^j] \qquad (5.6)
$$

in flat space. We see that this can be interpreted precisely as the general form given in eqs. (5.2) and (5.3). There is the background field $\phi(t, \Phi)$ which implicitly contains a nonabelian Taylor expansion, as in eq. (5.1). This multiplies an expression involving $D_t\Phi$ and $i[\Phi^i, \Phi^j]$, i.e. the zeroth moment. Of course, all of the linear couplings derived from the D-brane action will have this same general form, and so there is agreement between two calculations at this general level. One must make a detailed comparison of the zeroth moments then, to determine if there is in fact a precise agreement between the two calculations. In the present case, it is straightforward to show that the expansion of the square root in eq. (5.5) in fact yields precisely the dilaton zeroth moment in eq. (5.4). Rather than repeat the lengthy formula above, let us describe the expansion by saying that it yields: (i) at zero order, simply 1, as in eq. (5.4), (ii) at second order, the standard low energy kinetic and potential terms for the nonabelian scalar (up to an overall sign), as in eq. (5.4), and (iii) at fourth order, the dimensional reduction of the stringy corrections appearing in the d=10 SYM theory at fourth order in field strengths [46, 18], again as in eq. (5.4) — that the terms above provide the desired dimensionally reduced is manifest in the notation of ref. [20]. We should also comment that all of the quantities in the calculations of ref. [20] are dimensionless, i.e. they have set $\lambda = 1$. To restore the standard engineering dimensions of our notation, one would set: $X^i \to \sqrt{\lambda}\Phi^i$, $\partial_{x^i} \to \sqrt{\lambda}\partial_{x^i}$, $s \to t/\sqrt{\lambda}$, and $R \to R/\sqrt{\lambda}$. Then the agreement between eqs. (5.2) and (5.5) is completed by matching $R^{-1} = T_0$, which agrees with the standard duality between type-IIA superstring theory and $D = 11$ M-theory where $R = g\ell_s$ [1, 44]. In fact, the D-brane calculation indicates that in fact there are no additional commutator corrections at fourth order in the zeroth moment of the dilaton coupling (5.4). Further the expansion of the D-brane action could be carried out to the next order, where we know there are extra commutator corrections but these have been explicitly calculated [40]. Hence in principle, one could use these results to calculate the $O(v^6)$ corrections to eq. (5.4). Of course, this is only the next term in an infinite series, and so there would still be unknown terms at $O(v^8)$.

At this point, we come to the first of the aforementioned caveats. In the D-brane calculation, the higher moments of the dilaton coupling which are implicit in eq. (5.5) are actually ambiguous, since we have as yet not provided a specific prescription for implementing the gauge trace. On the other hand, there is no such ambiguity in the matrix theory calculation which comes with a specific symmetrized trace prescription. Hence matrix theory guides us to the appropriate prescription: the gauge trace takes a symmetrized average over all orderings of F_{ab}, $D_a\Phi^i$, $i[\Phi^i, \Phi^j]$, and as well the individual Φ^i appearing in the functional dependence of the background fields. We adopt this as our trace prescription in both the Born-Infeld (3.9) and Chern-Simons (4.14) actions. This will ensure the agreement of all of higher moments in the linear couplings to those calculated from matrix theory [20]. Note that we are proposing that this prescription should be extended to all of the nonlinear couplings with background fields as well. This maximally symmetrized trace prescription also agrees with the requirements that we outlined for the trace in the previous two sections — namely, it should be symmetric in F_{ab}, $D_a\Phi^i$ and $i[\Phi^i, \Phi^j]$, following ref. [18], and also that the individual components of background fields should commute under the trace.

Now let us consider the linear coupling of the RR five-form, which Taylor and van Raamsdonk [20] write as

$$\int ds \sum_{n=0}^{\infty} \frac{1}{n!} \left(\partial_{x^{i_1}} \cdots \partial_{x^{i_n}} \tilde{C}^{(3)}_{\mu\nu\lambda\rho\sigma} \right) I_4^{\mu\nu\lambda\rho\sigma(i_1 \cdots i_n)}, \tag{5.7}$$

where their notation indicates that this potential generates the field strength that is dual to that of the RR three-form. Now the various moments again have the form

$$I_4^{\mu\nu\lambda\rho\sigma(i_1 \cdots i_n)} = \mathrm{STr}(I_4^{\mu\nu\lambda\rho\sigma}, X^{i_1}, \ldots, X^{i_n}) \,. \tag{5.8}$$

As the zeroth moment now carries an antisymmetric set of spacetime indices there are two distinct cases depending on whether or not one of the indices coincides with the world-volume direction s. Taylor and van Raamsdonk [20] derived that

$$I_4^{sijkl} = -\frac{3}{2R}\left([X^{[i}, X^j][X^k, X^{l]}] + O(v^4)\right), \tag{5.9}$$

$$I_4^{ijklm} = 6M^{+ijklm} - \frac{15}{2R}\left(D_s X^{[i}[X^j, X^k][X^l, X^{m]}] + O(v^5)\right). \tag{5.10}$$

The $[\cdots]$ enclosing the indices indicates that the expressions are completely antisymmetrized, and as usual, all of the fermion contributions have been ignored. In the D-brane action (4.15), the linear coupling to this RR potential is

$$-\frac{\lambda^2\mu_0}{2}\int dt\,\mathrm{Tr}\,P\left[(\mathrm{i}_\Phi\mathrm{i}_\Phi)^2 C^{(5)}\right] = \tag{5.11}$$
$$= -\frac{\lambda^2\mu_0}{8}\int dt\,\mathrm{Tr}\left[C^{(5)}_{tijkl}[\Phi^{[i},\Phi^j][\Phi^k,\Phi^{l]}] + \lambda C^{(5)}_{ijklm} D_t\Phi^{[i}[\Phi^j,\Phi^k][\Phi^l,\Phi^{m]}]\right].$$

Comparing these expressions, one finds that there is precise agreement between the explicit couplings as before provided that the potentials in the two calculations are related as: $C^{(5)} = 60\,\tilde{C}^{(3)}$. Further the D-brane results suggest that there are no higher order corrections in eqs. (5.9) and (5.10), or to the zeroth moments of any of the RR potentials calculated in ref. [20].

At this point, the reader should think that we have concluded too quickly that there is a precise agreement in the previous calculation, as we have not commented on the contribution M^{+ijklm} in eq. (5.10). This brings us to the second of the caveats that we mentioned at the outset above. Above, of course, we only find a precise agreement if $M^{+ijklm} = 0$. In matrix theory, M^{+ijklm} is the transverse 5-brane charge, and determining this operator remains a famous unresolved puzzle. So here through the D-brane action, string theory is making a definite prediction as to the nature of this operator, namely, $M^{+ijklm} = 0$. This result is also required for the agreement of other couplings, as well. The matrix theory calculation [20] includes couplings to the six-form potential which would be the dual of the Neveu-Schwarz two form. The zeroth moments in this case are given by M^{+ijklm} and M^{ijklmn}, both of which remain undetermined in matrix theory. In the D-brane action, (3.9) and (4.14), there are simply no such couplings, and so agreement requires both $M^{+ijklm} = 0$ and $M^{ijklmn} = 0$. Actually, here and above, we have only really shown that what vanishes is the part of these operators which is independent of the world-volume fermions. However, it seems clear that introducing the fermions in a supersymmetric action will not create any new couplings to $\tilde{B}_{\mu\nu\lambda\rho\sigma\tau}$.

6. Dielectric-branes

In this section, we begin an investigation the physical effects arising from the new nonabelian interactions in the full world-volume action, i.e. the sum of eqs. (3.9) and (4.14). In particular, here we will focus on the potential for the nonabelian scalars. To begin, consider this potential for Dp-branes in a flat space background, i.e. $G_{\mu\nu} = \eta_{\mu\nu}$ with all other fields vanishing. In this case, the entire scalar potential originates in the Born-Infeld term (3.9) as

$$V = T_p \ \mathrm{Tr}\sqrt{\det(Q^i{}_j)} = NT_p + \frac{T_p\lambda^2}{4}\,\mathrm{Tr}([\Phi^i, \Phi^j]\,[\Phi^j, \Phi^i]) + \ldots \tag{6.1}$$

At low energies, the discussion focuses on the leading nontrivial contribution above

$$V_{SYM} = -\frac{T_p\lambda^2}{4}\,\mathrm{Tr}([\Phi^i, \Phi^j]\,[\Phi^i, \Phi^j])\,, \tag{6.2}$$

which corresponds to the potential for ten-dimensional U(N) super Yang-Mills theory reduced to $p+1$ dimensions. A nontrivial set of extrema of this potential (and the full expression in eq. (6.1)) is given by taking the $9-p$ scalars as constant commuting matrices, i.e.

$$[\Phi^i, \Phi^j] = 0 \tag{6.3}$$

for all i and j. Since they are commuting, the Φ^i may be simultaneously diagonalized to give

$$\Phi^i = \begin{pmatrix} x_1^i & 0 & 0 & \ddots \\ 0 & x_2^i & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ \ddots & 0 & 0 & x_N^i \end{pmatrix}. \tag{6.4}$$

Here, the eigenvalues are interpreted as the separated positions in the transverse space of N fundamental Dp-branes — see, for example, ref. [34]. This solution reflects the fact that a system of N parallel Dp-branes is supersymmetric, and they can rest in static equilibrium with arbitrary separations in the transverse space [3].

We have seen that going from flat space to a general background introduces a vast number of new nonderivative scalar interactions in the world-volume action. These arise from the functional dependence of the background fields on the scalars, and the appearance of extra nonabelian commutators. Various physical effects governed by both of these types of interactions have previously been considered. For example, the former interactions are important in determining the dynamics of test Dp-branes in nontrivial backgrounds, e.g., [52]. In the context of matrix theory, the role of the second set of interactions that was emphasized was to allow Dp-branes supporting noncommuting configurations of the scalars to act a sources for RR potentials with form degree greater than $p+1$, e.g., [20, 44, 45, 48].

Here we want to consider another interesting physical effect — that is the D-brane analog of the dielectric effect in ordinary electromagnetism. When Dp-branes are placed in a nontrivial background field for which the Dp-branes would normally be regarded as neutral, e.g., nontrivial $F^{(n)}$ with $n > p+2$, new terms will be induced in the scalar potential, and generically one should expect that there will be new extrema beyond those found in flat space, i.e. eq. (6.3). In particular, there can be nontrivial extrema with noncommuting expectation values of the Φ^i, e.g., with $\mathrm{Tr}\,\Phi^i = 0$ but $\mathrm{Tr}(\Phi^i)^2 \neq 0$. This would correspond to the external field "polarizing" the Dp-branes to expand into a (higher dimensional) noncommutative world-volume geometry. This is the analog of the familiar electromagnetic process where an external field may induce a separation of charges in neutral materials. In the latter, the polarized material will then carry then an electric dipole and possibly higher multipoles. The D-brane analog of the latter is that when the world-volume theory is at a noncommutative extremum, the nontrivial expectation values of the scalars will cause the Dp-branes to act as a source for new background fields. To make these ideas explicit, we will illustrate the process with a simple example below. Below, we consider N D0-branes in a constant background RR field $F^{(4)}$, i.e. the field strength associated with D2-brane charge. We will find that the D0-branes expand into a noncommutative configuration which represents the spherical bound state of a D2-brane and N D0-branes.

6.1 Dielectric D0-branes

Consider a collection of N D0-branes in a constant background RR four-form field strength

$$F^{(4)}_{tijk} = \begin{cases} -2f\varepsilon_{ijk} & \text{for } i,j,k \in \{1,2,3\} \\ 0 & \text{otherwise} \end{cases} \tag{6.5}$$

Note that f carries the dimensions (length)$^{-1}$, and we are assuming $\sigma^0 = t$, with the static gauge condition. Since $F^{(4)} = dC^{(3)}$, in order to construct the scalar potential, we must consider the coupling of the D0-branes to the RR three-form potential,

$$\begin{aligned} i\lambda\,\mu_0 \int \mathrm{Tr}\; P\left[\mathrm{i}_\Phi \mathrm{i}_\Phi C^{(3)}\right] &= \\ &= i\lambda\,\mu_0 \int dt\; \mathrm{Tr}\left[\Phi^j\Phi^i\left(C^{(3)}_{ijt}(\Phi,t) + \lambda C^{(3)}_{ijk}(\Phi,t)\,D_t\Phi^k\right)\right] \\ &= i\lambda\mu_0 \int dt\;\mathrm{Tr}\left[\Phi^j\Phi^i\left(C^{(3)}_{ijt}(t) + \lambda\Phi^k\partial_k C^{(3)}_{ijt}(t) + \frac{\lambda^2}{2}\Phi^l\Phi^k\partial_l\partial_k C^{(3)}_{ijt}(t) + \cdots\right.\right. \\ &\qquad\left.\left. + \lambda C^{(3)}_{ijk}(t)\,D_t\Phi^k + \lambda^2\Phi^l\partial_l C^{(3)}_{ijk}(t)\,D_t\Phi^k + \cdots\right)\right], \end{aligned} \tag{6.6}$$

where in the second line, we are explicitly introducing the nonabelian Taylor series expansion (5.1) of the RR potential. As noted below eq. (4.16), the quadratic term containing $C^{(3)}_{ijt}(t)$ vanishes. Focusing on the terms that are cubic in the scalar fields, we have

$$i\lambda^2\mu_0\int dt\;\mathrm{Tr}\left(\Phi^j\Phi^i\left[\Phi^k\partial_k C^{(3)}_{ijt}(t) + C^{(3)}_{ijk}(t) D_t\Phi^k\right]\right) = \frac{i}{3}\lambda^2\mu_0\int dt\;\mathrm{Tr}\left(\Phi^i\Phi^j\Phi^k\right) F^{(4)}_{tijk}(t)\,. \tag{6.7}$$

Here, the second term in the first line was integrated by parts to produce the final expression. This final form might have been anticipated since one should expect that the induced potential can only depend on gauge invariant expressions of the background field. Given that for simplicity we are considering a constant background $F^{(4)}$, this interaction (6.7) is the only term that need be considered. All of the higher order terms implicit in eq. (6.6) will give rise to potential terms depending on derivatives of the four-form field strength, and hence vanish. Combining eq. (6.7) with the leading order Born-Infeld potential (6.2) yields the scalar potential of interest for the present problem

$$V(\Phi) = -\frac{\lambda^2 T_0}{4}\,\mathrm{Tr}([\Phi^i,\Phi^j]^2) - \frac{i}{3}\lambda^2\mu_0\,\mathrm{Tr}\left(\Phi^i\Phi^j\Phi^k\right)F^{(4)}_{tijk}(t)\,. \tag{6.8}$$

Substituting the background field (6.5) and demanding $\delta V(\Phi)/\delta\Phi^i = 0$ yields the equation for extrema

$$0 = [[\Phi^i,\Phi^j],\Phi^j] + i\,f\varepsilon_{ijk}[\Phi^j,\Phi^k]\,. \tag{6.9}$$

Note that commuting matrices (6.3) still solve this equation, and at this extremum describing separated D0-branes, the value of the potential is simply $V_0 = 0$.

As an ansatz for a noncommuting solution, let us consider constant scalars satisfying

$$[\Phi^i,\Phi^j] = 2i\hat{R}\,\varepsilon_{ijk}\Phi^k\,. \tag{6.10}$$

Substituting this ansatz into eq. (6.9) yields a solution for

$$\hat{R} = \frac{f}{2}\,. \tag{6.11}$$

Hence we have nontrivial solutions of eq. (6.9)

$$\Phi^i = \frac{f}{2}\,\alpha^i \tag{6.12}$$

where α^i are any $N\times N$ matrix representation of the SU(2) algebra

$$[\alpha^i,\alpha^j] = 2i\,\varepsilon_{ijk}\,\alpha^k\,. \tag{6.13}$$

For the moment, let us focus on the irreducible representation for which one finds

$$\mathrm{Tr}[(\alpha^i_{\scriptscriptstyle N})^2] = \frac{N}{3}(N^2-1) \quad \text{for } i=1,2,3. \tag{6.14}$$

Now evaluating the value of the potential (6.8) for this new solution yields

$$V_N = -\frac{T_0\lambda^2 f^2}{6}\sum_{i=1}^{3}\mathrm{Tr}[(\Phi^i)^2] = -\frac{\pi^2\ell_s^3 f^4}{6g}N(N^2-1)\,. \tag{6.15}$$

Hence the noncommutative solution solution has lower energy than the solution of commuting matrices, and so the latter configuration separated D0-branes is unstable towards condensing out into this noncommutative solution.

Of course, the irreducible $N \times N$ representation provides only one solution for eq. (6.12). One can compose lower dimensional representations (including the trivial 1×1 representation, 0) in a direct sum to produce an alternative $N\times N$ representation in eq. (6.12). However, such reducible representations would yield $\mathrm{Tr}[(\alpha^i)^2]$ which is less than that for the irreducible representation (6.14). Hence, for the reducible representations, one always finds

$$V_N < V_r \leq V_0 = 0\,. \tag{6.16}$$

Hence these noncommutative configurations, based on reducible representations, would correspond to intermediate unstable extrema of the potential (6.8), and it appears that the irreducible representation describes the ground state of the system (and we refer to it as such in the following).

Given this ground state, one should note that since both terms in the equations of motion involve commutators, one can still modify the solution by

$$\Phi^i = \frac{f}{2}\,\alpha^i_{\scriptscriptstyle N} + x^i\, I_{\scriptscriptstyle N}\,, \tag{6.17}$$

where $I_{\scriptscriptstyle N}$ corresponds to the N×N identity matrix. This modification corresponds to shifting the center of mass of the D0-branes to

$$\frac{1}{N}\,\mathrm{Tr}(\Phi^i) = x^i \tag{6.18}$$

and of course, the value of the potential remains unchanged as in eq. (6.15). For the purposes of the following discussion we will assume that $x^i = 0$ without loss of generality.[11]

Geometrically, one can recognize the algebra (6.10) as that corresponding to the noncommutative or fuzzy two-sphere [53, 54]. In the context of matrix theory, this noncommutative geometry was discussed in ref. [55]. The extent of the noncommutative world-volume can be measured as

$$R = \lambda\left(\sum_{i=1}^{3}\frac{\mathrm{Tr}[(\Phi^i)^2]}{N}\right)^{1/2} = \pi\ell_s^2 f N\sqrt{1-\frac{1}{N^2}} \tag{6.19}$$

for the ground state solution. For later convenience, we let us define the radius $R_0 \equiv \pi\ell_s^2 f N$, which for large values of N gives essentially the physical size of the noncommutative geometry. One may infer from the matrix model construction of Kabat and Taylor [55] that the noncommutative solution actually represents a spherical D2-brane with N D0-branes bound to it. While such a configuration carries no net D2-brane charge, there would be a “dipole” coupling due to the seperation of

[11] If α^i is reducible, one may make similarly independent shifts for each of the irreducible representations in the direct sum.

oppositely oriented surface elements of membrane. The precise form of this coupling can be calculated by substituting the noncommutative scalar solution (6.12) into the world-volume interaction (6.7). Using the ground state solution, the result can be written as

$$-\frac{R_0^3}{3\pi g\ell_s^3}\left(1-\frac{1}{N^2}\right)\int dt\, F^{(4)}_{t123}\,. \tag{6.20}$$

In fact, this is only the leading source term that would be generated by substituting the noncommutative solution into the full expansion in eq. (6.6) — that is, there will also be a series of higher order "multipole" couplings, as well.

6.2 Dual D2-branes

In the ground state solution above, the D0-branes condense out in a noncommutative configuration, which we argued represents a bound state of a spherical D2-brane and N D0-branes. It is interesting to investigate to what extent these configurations can be matched in the dual formulation of the same system. That is, an abelian three-dimensional world-volume theory of a D2-brane in which a flux of the U(1) gauge field strength represents the N bound D0-branes. To parallel the same physics as above, we consider this test D2-brane system in a constant background RR four-form field strength (6.5), and look for a stable static configuration resembling that above.

To simplify the calculations, we write the flat space metric using spherical polar coordinates to replace $x^{1,2,3}$, i.e.

$$ds^2=-dt^2+dr^2+r^2\left(d\theta^2+\sin^2\theta\, d\phi^2\right)+\sum_{i=4}^{9}(dx^i)^2\,. \tag{6.21}$$

In static gauge, we choose the world-volume coordinates on the D2-brane to be $\sigma^0=t,\sigma^1=\theta,\sigma^3=\phi$. We will look for a static solution of the form

$$r=R \tag{6.22}$$

(and $x^i=0$ for $i=4,\ldots,9$). The background four-form (6.5) should also be adapted to the spherical polar coordinates

$$F^{(4)}_{tr\theta\phi}=-2fr^2\sin\theta\;\longrightarrow\;C^{(3)}_{t\theta\phi}=\frac{2}{3}fr^3\sin\theta\,, \tag{6.23}$$

where we have made a convenient gauge choice for three-form potential. The last ingredient required before proceeding to the action is the U(1) field strength describing the N bound D0-branes,

$$F_{\theta\phi}=\frac{N}{2}\sin\theta\,. \tag{6.24}$$

One can confirm the normalization of this flux by considering the induced coupling to the RR one-form in the Chern-Simons action (1.3). A simple calculation shows that

$$\lambda\mu_2\int C^{(1)}\,F=N\mu_0\int dt\, C^{(1)}_t\,, \tag{6.25}$$

i.e. the (monopole) coupling to N fundamental D0-branes is reproduced.

With the above results in hand, we turn to the action which in the present case reduces to

$$S = -T_2 \int dt\, d\theta\, d\phi \sqrt{-\det(P[G]_{ab} + \lambda F_{ab})} + \mu_2 \int P[C^{(3)}] \,. \tag{6.26}$$

Note that for these calculations, we have a single D2-brane and so the world-volume theory is abelian. Eq. (6.26) fixes the lagrangean density $\mathcal{L}$, and so for the static trial solution (6.22), the potential energy is given by

$$\begin{aligned} V(R) &= -\int d\theta\, d\phi\, \mathcal{L} \\ &= 4\pi T_2 \left(\sqrt{R^4 + \frac{\lambda^2 N^2}{4}} - \frac{2f}{3} R^3 \right) \\ &= N T_0 + \frac{2T_0}{\lambda^2 N} R^4 + \cdots - \frac{4T_0}{3\lambda} f R^3 \,. \end{aligned} \tag{6.27}$$

In the last line, we have expanded the square root for $2R^2/\lambda N \ll 1$, only keeping the first two terms in the expansion explicitly, and we have substituted $T_0 = 2\pi\lambda T_2$ everywhere.

The first thing that we note is that the constant term in the potential energy corresponds precisely to the rest energy of N D0-branes. Thus this precisely matches the same term in the nonabelian D0-brane calculation in the previous section — this is the “trivial” constant term in eq. (6.1), which was then overlooked in eq. (6.8). Keeping only the leading order terms given in eq. (6.27), there are two extrema:

$$\begin{aligned} &R = 0 && \text{with } V - NT_0 = 0 \,, \\ &R = R_0 = \pi \ell_s^2 f N && \text{with } V - NT_0 = -\frac{\pi^2 \ell_s^3 f^4}{6g} N^3 \,. \end{aligned} \tag{6.28}$$

The first extrema is an unstable inflection point in the potential, while the second is a stable spherical configuration. Comparing the equilibrium radius with eq. (6.19), we see that this calculation reproduces the noncommutative ground state radius up to $1/N^2$ corrections. Similarly the shift in the potential energy reproduces the result for the D0-brane result (6.15), again up to $1/N^2$ corrections.

Finally, just as in the previous section, this spherical D2-brane configuration carries no net D2-brane charge. However, because the membrane is supported at a finite radius, there is a finite dipole coupling:

$$\mu_2 \int P[C^{(3)}] = -\frac{R_0^3}{3\pi g \ell_s^3} \int dt\, F^{(4)}_{t123} + \cdots \,, \tag{6.29}$$

where we have only retained the leading term, and expressed the result in cartesian coordinates, to compare to eq. (6.20). Once again, the two calculations agree up to $1/N^2$ corrections. It would be interesting to compare the higher order multipole couplings in the two different frameworks.

JHEP12(1999)022

7. Discussion

Using T-duality, we have provided a straightforward construction of a nonabelian world-volume action describing the dynamics of N Dp-branes in general background fields. The action is composed of two parts, the Born-Infeld action

$$S_{BI} = -T_p \int d^{p+1}\sigma \, \mathrm{STr}\left(e^{-\phi} \sqrt{-\det\left(P\left[E_{ab} + E_{ai}(Q^{-1}-\delta)^{ij}E_{jb}\right] + \lambda F_{ab}\right)\det(Q^i{}_j)}\right), \tag{7.1}$$

with

$$E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}\,, \qquad \text{and} \qquad Q^i{}_j \equiv \delta^i{}_j + i\lambda\,[\Phi^i, \Phi^k]\,E_{kj}\,, \tag{7.2}$$

and the Chern-Simons action

$$S_{CS} = \mu_p \int \mathrm{STr}\left(P\left[e^{i\lambda \mathrm{i}_\Phi \mathrm{i}_\Phi}\left(\sum C^{(n)}\, e^{B}\right)\right] e^{\lambda F}\right). \tag{7.3}$$

One of the most striking features of this result is that in the nonabelian action, Dp-branes couple to RR potentials with a form degree greater than the dimension of the world-volume. These couplings arise through inner products with the nonabelian scalars, and vanish for the abelian theory (1.3).

In both eqs. (7.1) and (7.3), the gauge trace is indicated with $\mathrm{STr}(\cdots)$ to indicate its symmetrized nature. The precise prescription which we propose is that inside the trace one takes a symmetrized average over all orderings of the F_{ab}, $D_a\Phi^i$, $i[\Phi^i,\Phi^j]$, and also the individual Φ^i appearing in the functional dependence of the background fields. This trace prescription was adopted from matrix theory, and was required to provide a precise agreement of the linear background couplings for D0-branes derived from our action above, and those derived from matrix theory [20]. This symmetrized trace coincides with Tseytlin's proposal for the nonabelian Born-Infeld gauge theory, i.e. eq. (7.1) in flat space. Investigations of this proposal have shown that it requires additional corrections involving commutators of field strengths at sixth order [39, 40]. However, for certain configurations, the flat space Born-Infeld action seems to capture the full physics of the nonabelian Yang-Mills fields [41, 42]. In any event, one should expect that our nonabelian action will also require higher order commutator corrections in generic situations. It would be of interest to determine the precise limitations of our action, or rather to determine the situations where it does reliably describe the physics.

One limitation of our analyses is that we have considered only the bosonic fields. There have been numerous investigations into constructing supersymmetric world-volume actions for D-branes [56]. Given these results, it would, of course, be interesting to extend the actions constructed here to incorporate the interactions of the world-volume fermions. In this direction, we note that the matrix theory construction of the linear background couplings for the D0-brane [20] included the fermions, and so by T-duality, it can be used to extend these couplings to arbitrary Dp-branes [51].

Since these world-volume actions, (7.1) and (7.3), are low energy string actions, there is an infinite series of higher order α' interactions. The Born-Infeld term (7.1) already includes an infinite set of such interactions involving powers of F_{ab}, $D_a\Phi^i$ and $i[\Phi^i, \Phi^j]$. In the abelian theory, the action (1.1) properly resums all the α' corrections when the derivatives of the field strength and second derivatives of the scalars vanish [7]. One still expects further corrections involving higher derivatives of the world-volume fields. For the nonabelian theory, the distinction between derivatives and field strengths becomes less clear, e.g., the higher order commutator corrections [39], mentioned above, can be rewritten as higher derivative corrections. The results of ref. [42] are an indication that the action (7.1) does properly resum the α' corrections for constant self-dual or supersymmetric field strengths.

By introducing general background fields in either the abelian or nonabelian actions, one is implicitly also including an infinite series of higher order α' corrections through the functional dependence of these fields on the transverse directions, as illustrated in eq. (5.1). There are still further higher derivative corrections involving the background fields that are independent of the transverse fields [57, 58, 59]. The focus of most investigations in this direction have been on anomalous curvature couplings appearing in the Chern-Simons action [58, 59]. These couplings must also appear in the nonabelian action (7.3), but it would be interesting to determine their precise form, in light of the commutator corrections discovered for the nonabelian scalars.

A further comment on these anomalous couplings is that it seems the interactions appearing in the literature [58, 59] must be incomplete. Recall that the guiding principle in constructing eqs. (7.1) and (7.3) was that the actions should be consistent with T-duality. The known anomalous curvature couplings [58, 59] only consider the curvature of the metric connection, but since T-duality exchanges components of the metric and the Neveu-Schwarz two-form, these couplings will not be compatible with T-duality. The consistent couplings will most likely involve the generalized curvatures of a torsionful connection constructed by combining the metric spin-connection and the Neveu-Schwarz three-form field strength, e.g., [60]. Recall that these couplings arise from the anomalous transformation properties of the world-volume spinors. So one piece of evidence in favor of generalized curvatures appearing in these interactions is that such a torsion term appears in the full connection of the super Yang-Mills fermions of the D9-brane, e.g., [61]. This makes understanding the couplings of the world-volume fermions for general Dp-branes an even more interesting task.

Before leaving our discussion of the action, (7.1) and (7.3), we remark again that the comparison between this action and the matrix theory action in section 5 made a definite prediction about that transverse five-brane charges. That is, these elusive charges should be exactly zero, i.e. $M^{+ijklm} = 0 = M^{ijklmn}$. One may note that this string theory prediction is in accord with the arguments in ref. [45] that transverse five-branes should not exist in the infinite momentum frame of matrix theory. On the other hand, an implicit matrix theory construction for transverse five-branes has been suggested using the S-duality of four-dimensional super Yang-Mills theory [62]. This is related to the observation that while D-branes have no obvious couplings to $\tilde{B}_{\mu\nu\lambda\rho\sigma\tau}$, the S-duality of type-IIB superstring theory interchanges D5-branes and NS5-branes. Hence the correct conclusion here seems to be that perturbatively M^{+ijklm} and M^{ijklmn} must vanish, but that these couplings may be generated nonperturbatively at strong coupling.

Given our nonabelian action, in section 6, we discussed the D-brane analog of the dielectric effect. When Dp-branes are placed in a nontrivial background field for which the Dp-branes would normally be regarded as neutral, e.g., nontrivial $F^{(n)}$ with $n > p+2$, new terms will be induced in the scalar potential, and new noncommutative extrema can be generated. At such a noncommutative extremum, the nontrivial expectation values of the scalars will cause the Dp-branes to act as a multipole source for new background fields. These ideas were explicitly illustrated by considering a system of N D0-branes in a constant background RR field $F^{(4)}$, and it was shown the D0-branes expand into a noncommutative two-sphere, which carried a dipole coupling for $F^{(4)}$.

One should note that there are similar couplings to the Neveu-Schwarz two-form implicit in the Born-Infeld action. From the expansion of $\sqrt{\det(Q)}$, one finds an interaction of the form

$$\frac{i}{3}\lambda^2 T_0 \int dt\ \mathrm{Tr}\left(\Phi^i\Phi^j\Phi^k\right) H_{ijk}(t)\,. \tag{7.4}$$

Hence the noncommutative ground state, in section 6.1 for which $\mathrm{Tr}\left(\Phi^i\Phi^j\Phi^k\right) \neq 0$, also acts as a source of the B field with

$$-\frac{R_0^3}{3\pi g\ell_s^3}\left(1-\frac{1}{N^2}\right)\int dt\, H_{123}\,. \tag{7.5}$$

This coupling is perhaps not so surprising, as the noncommutative ground state represents the bound state of a spherical D2-brane and N D0-branes. Explicit supergravity solutions describing such bound states with a planar geometry are known [26, 63], and carry a long-range H field with the same profile as the RR field strength $F^{(4)}$. One can also derive this coupling from the dual D2-brane formulation used in section 6.2. Furthermore, we observe that the presence of this coupling (7.4) means that we would find an analogous dielectric effect if the N D0-branes were placed in a constant background H field.

In section 6.2, we compared the results of the nonabelian D0-brane calculations to those derived from an abelian D2-brane. The remarkable aspect of this comparison was the degree to which the results agreed, at least in the limit of large N. This agreement becomes even more striking when we compare the full potential for the radial position. For the dual D2-brane, this potential appears in the second line of eq. (6.27), and we rewrite it here as

$$V_2(R) = NT_0 \left(\sqrt{1 + \frac{4R^4}{\lambda^2 N^2}} - \frac{4f}{3\lambda N} R^3 \right) . \tag{7.6}$$

To calculate the same for the D0-branes, we begin with the ansatz in eq. (6.10)

$$\Phi^i = \hat{R}\, \alpha^i_{{}_N} , \tag{7.7}$$

where we will consider only the irreducible N×N representation of the SU(2) generators, but we do not fix $\hat{R}$ with the equations of motion (6.9). The corresponding physical radius of the corresponding noncommutative two-sphere is given by

$$R^2 = \frac{\lambda^2}{N} \sum_{i=1}^{3} \mathrm{Tr}[(\Phi^i)^2] = \lambda^2 \hat{R}^2 (N^2 - 1) . \tag{7.8}$$

Now the Born-Infeld potential is determined by the determinant of the matrix Q defined in eq. (7.2), which in the present case becomes

$$\begin{aligned} \det(Q_{ij}) &= \det\left(\delta_{ij} - 2\lambda \hat{R}^2 \varepsilon_{ijk} \alpha^k_{{}_N}\right) \\ &= 1 + 4\lambda^2 \hat{R}^4 \sum_{i=1}^{3} (\alpha^i_{{}_N})^2 - 8\lambda^3 \hat{R}^6 \left(\alpha^3_{{}_N} \alpha^1_{{}_N} \alpha^2_{{}_N} - \alpha^2_{{}_N} \alpha^3_{{}_N} \alpha^1_{{}_N}\right) . \end{aligned} \tag{7.9}$$

Now we make two useful observations: First, the ordering of the matrices in the last two terms of the determinant is ambiguous. However, in determining the potential, this expression will appear inside the symmetrized trace, and so by the symmetric averaging there, the contributions of these two terms will always cancel. Hence, we can drop them in the following calculations. The other observation is that the SU(2) generators satisfy the relation

$$\sum_{i=1}^{3} (\alpha^i_{{}_N})^2 = (N^2 - 1)\, I_{{}_N} , \tag{7.10}$$

i.e. this is the quadratic Casimir of the group. Hence the two terms contributing in eq. (7.9) are both proportional to the identity, and so the gauge trace simply yields an overall factor[12] of N. Putting all of these results together, one finds that the full potential arising from the nonabelian D0-brane action may be written

$$V_0(R) = NT_0 \left(1 - \frac{1}{N^2}\right)^{-1/2} \left(\sqrt{1 - \frac{1}{N^2} + \frac{4R^4}{\lambda^2 N^2}} - \frac{4f}{3\lambda N} R^3 \right) . \tag{7.11}$$

One may confirm that if V_0 is expanded for small $4R^4/\lambda N^2$, one recovers precisely eq. (6.19) as the position of the minimum. Comparing the two potentials, (7.6) and (7.11), we now see that there will be good agreement for the two calculations at large N using the entire potential, not just with the leading order expansion used in section 6.

[12] Actually this only captures the leading contribution at each order in R — see ref. [66] for details.

JHEP12(1999)022

One way to understand this remarkable agreement is to consider the commutator of the scalars expressed in terms of the physical radius (7.8). In terms of dimensionless matrices, the commutator becomes

$$\left[\,\ell_s\Phi^i, \ell_s\Phi^j\,\right] = \frac{i}{\pi\sqrt{N^2-1}}\frac{R}{\ell_s}\,\varepsilon_{ijk}\,\ell_s\Phi^k\,. \tag{7.12}$$

Here we see that for fixed R and ℓ_s, the matrices become nearly commuting when N becomes large. Hence the nonabelian character of the solution diminishes and so it is not surprising that the D0-brane results match those derived from the abelian D2-brane theory.

Given the limitations of the nonabelian action discussed above, i.e. possible commutator corrections at higher orders in F, one would conclude that the nonabelian D0-brane calculations should only be trusted for small commutators. Naively, one would expect that this would translate into restricting the size of the noncommutative sphere to be small compared to the string scale. The true restriction is that the Taylor expansion of the square root in the potential (7.11) converge rapidly, and this only requires that $R \ll \sqrt{N}\ell_s$. Hence for large N, one may consider radii far larger than the string scale.

One may also argue that the radius must remain small since the low energy action, (7.1) and (7.3), is simply inapplicable if the D0-branes become separated by more than the string scale. That is, if the "low energy" modes have masses of the order of $1/\ell_s$, then one could not ignore oscillator excitations in determining the physics of the open string sector. While the latter is true, the radius R is not the correct scale to characterize the separation of the D0-branes. Rather one should think of the D0-branes forming area elements of the noncommutative two-sphere [54, 55]. The typical size of the area elements is $4\pi R^2/N$, and so if as above $R \ll \sqrt{N}\ell_s$, the typical separations between D0-branes is still much less than the string scale, as required. The latter result can be confirmed by considering the frequency of perturbations about the extremum (6.19). One finds that $\omega^2 = n(n+1)f^2$ for large N where n is the angular quantum number of the fluctuation mode on the noncommutative sphere. Now if $R_0 \ll \sqrt{N}\ell_s$ then $f \ll 1/(\sqrt{N}\ell_s)$. Hence we see that the energy of the low energy excitations is far below the string mass scale, and so will be adequately described by the low energy world-volume action.

Being able to consider large R allows us to match onto the regime where the membrane action of the dual D2-brane calculations is also applicable. Of course, systems like the D2-brane carrying a large gauge field flux have recently been of interest for the connections to noncommutative gauge theories, e.g., [64, 65]. Following the analysis of ref. [64], one finds that the regime upon which we focussed above is precisely that in which the standard string coupling g is much smaller than the effective string coupling G_s for the theory described in terms of noncommutative variables, i.e.

$$g \simeq \frac{2R^2}{\lambda N} G_s \ll G_s \,. \tag{7.13}$$

Thus ordinary gauge theory provides the most efficient description of the D2-branes, i.e. the system is essentially abelian. It would be interesting to try to study the system in a regime where the noncommutative variables proved more efficient. Ultimately, one expects the D2-brane theory to give rise to the nonabelian D0-brane physics, but making the connection more precise should prove illuminating.

Still if the nonabelian D0- and abelian D2-brane calculations are simply dual formulations of the same physical system, then for large but finite N, the D2-brane framework should still capture the small but finite noncommutative properties found in the D0-brane calculations. Ref. [64] tells us that the difference in formulating the D2-brane theory in terms of an ordinary or noncommutative gauge theory is in the higher derivative corrections to the world-volume action. So perhaps the noncommutative aspects of the bound state system will only appear in the D2-brane formalism with the inclusion of certain higher derivative corrections to the abelian action, (1.1) and (1.3). Ordinarily such terms would be ignored on the basis that their contributions give only small corrections, but in the present case, it is precisely small $1/N$ corrections that are of interest. Thus we are advocating that resolving the discrepancies between the dual pictures requires determining corrections to *both* formalisms, e.g., higher order commutator corrections for the D0-branes, and higher derivative corrections for the D2-branes.

Finally, after arguing the merits of our calculations in section 6, we must still acknowledge one significant shortcoming — namely the supergravity background is not a consistent background solution of the type-IIA supergravity equations of motion. One could argue that this background might be approximated by placing the D0-branes in the asymptotic region far from a D2-brane, and focusing on a small region in which the gradients of the background fields can be neglected. One can not take seriously all of the details of the calculations. For example, the potentials (7.6) and (7.11) are in fact unstable since they are dominated by the R^3 term at large radius. However, this instability will only become apparent at radii of the order of $1/f$, but the gravitational effects of the total energy of the four-form enclosed in such a region will be of order 1. Hence the gravitational back-reaction cannot be neglected on this scale, and so we conclude that this instability is simply an artifact of not chosing a consistent solution for the background.

However, the calculations in section 6 were presented in the spirit of a toy calculation to illustrate the effects of the new nonabelian interactions appearing in the action, (7.1) and (7.3), and in particular to demonstrate the dielectric effect for D-branes. A more detailed examination of these issues for realistic backgrounds is in preparation [66].

Acknowledgments

This research was supported by NSERC of Canada and Fonds FCAR du Québec. I would like to acknowledge useful conversations with Neil Constable, Clifford Johnson, Aki Hashimoto, Joe Polchinski, Oyvind Tafjord, Washington Taylor and Mark van Raamsdonk. Research at the ITP, UCSB was supported by NSF Grant PHY94-07194.

Note added. While this paper was in the final stages of preparation, I was informed by Washington Taylor and Mark van Raamsdonk of their recent work [51], which has significant overlap with the material in sections 3 and 4.

References

[1] E. Witten, *String theory dynamics in various dimensions*, *Nucl. Phys.* **B 443** (1995) 85 [hep-th/9503124]; *Some comments on string dynamics*, hep-th/9507121.

[2] J. Polchinski, *Dirichlet-branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724 [hep-th/9510017].

[3] J. Polchinski, *Tasi lectures on D-branes*, hep-th/9611050; J. Polchinski, S. Chaudhuri and C.V. Johnson, *Notes on D-branes*, hep-th/9602052;
C.V. Johnson, *Etudes on D-branes*, hep-th/9812196.

[4] P. Hořava, *Background duality of open string models*, *Phys. Lett.* **B 231** (1989) 251; *Strings on world-sheet orbifolds*, *Nucl. Phys.* **B 327** (1989) 461.

[5] J. Dai, R.G. Leigh and J. Polchinski, *New connections between string theories*, *Mod. Phys. Lett.* **A 4** (1989) 2073.

[6] M.B. Green, *Space-time duality and dirichlet string theory*, *Phys. Lett.* **B 266** (1991) 325.

[7] C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, *Loop corrections to superstring equations of motion*, *Nucl. Phys.* **B 308** (1988) 221;
A. Abouelsaood, C.G. Callan, C.R. Nappi and S.A. Yost, *Open strings in background gauge fields*, *Nucl. Phys.* **B 280** (1987) 599.

[8] R.G. Leigh, *Dirac-Born-Infeld action from Dirichlet sigma model*, *Mod. Phys. Lett.* **A 4** (1989) 2767.

[9] M.R. Douglas, *Branes within branes*, hep-th/9512077.

[10] M. Li, *Boundary states of D-branes and Dy-strings*, *Nucl. Phys.* **B 460** (1996) 351 [hep-th/9510161];
M.B. Green, J.A. Harvey and G. Moore, *I-brane inflow and anomalous couplings on D-branes*, *Class. and Quant. Grav.* **14** (1997) 47 [hep-th/9605033].

[11] E. Witten, *Bound states of strings and p-branes*, *Nucl. Phys.* **B 460** (1996) 335 [hep-th/9510135].

[12] C.G. Callan Jr. and J.M. Maldacena, *Brane dynamics from the Born-Infeld action*, *Nucl. Phys.* **B 513** (1998) 198 [hep-th/9708147];
G.W. Gibbons, *Born-Infeld particles and Dirichlet p-branes*, *Nucl. Phys.* **B 514** (1998) 603 [hep-th/9709027];
P.S. Howe, N.D. Lambert and P.C. West, *The self-dual string soliton*, *Nucl. Phys.* **B 515** (1998) 203 [hep-th/9709014];
S. Lee, A. Peet and L. Thorlacius, *Brane-waves and strings*, *Nucl. Phys.* **B 514** (1998) 161 [hep-th/9710097];
R. Emparan, *Born-Infeld strings tunneling to D-branes*, *Phys. Lett.* **B 423** (1998) 71 [hep-th/9711106].

[13] M.R. Douglas, *D-branes and matrix theory in curved space*, *Nucl. Phys.* **68** *(Proc. Suppl.)* (1998) 381 [hep-th/9707228]; *D-branes in curved space*, *Adv. Theor. Math. Phys.* **1** (1998) 198 [hep-th/9703056];
M.R. Douglas, A. Kato and H. Ooguri, *D-brane actions on Kähler manifolds*, *Adv. Theor. Math. Phys.* **1** (1998) 237 [hep-th/9708012].

[14] P. Brax and T. Wynter, *Limits of matrix theory in curved space*, *Nucl. Phys.* **B 546** (1999) 182 [hep-th/9806176];
F. Lizzi, N.E. Mavromatos and R.J. Szabo, *Matrix sigma-models for multi D-brane dynamics*, *Mod. Phys. Lett.* **A 13** (1998) 829 [hep-th/9711012];
N.E. Mavromatos and R.J. Szabo, *Matrix D-brane dynamics, logarithmic operators and quantization of noncommutative spacetime*, *Phys. Rev.* **D 59** (1999) 104018 [hep-th/9808124].

[15] C.M. Hull, *Matrix theory, U-duality and toroidal compactifications of M-theory*, *J. High Energy Phys.* **10** (1998) 011 [hep-th/9711179];
H. Dorn, *Nonabelian gauge field dynamics on matrix D-branes*, *Nucl. Phys.* **B 494** (1997) 105 [hep-th/9612120].

[16] M.R. Garousi and R.C. Myers, *World-volume interactions on D-branes*, *Nucl. Phys.* **B 542** (1999) 73 [hep-th/9809100].

[17] M.R. Garousi and R.C. Myers, in preparation.

[18] A.A. Tseytlin, *On non-abelian generalisation of the Born-Infeld action in string theory*, *Nucl. Phys.* **B 501** (1997) 41 [hep-th/9701125].

[19] A.A. Tseytlin, *Born-Infeld action, supersymmetry and string theory*, hep-th/9908105.

[20] I.W. Taylor and M.V. Raamsdonk, *Multiple D0-branes in weakly curved backgrounds*, *Nucl. Phys.* **B 558** (1999) 63 [hep-th/9904095].

[21] E. Bergshoeff, C. Hull and T. Ortin, *Duality in the type-II superstring effective action*, *Nucl. Phys.* **B 451** (1995) 547 [hep-th/9504081].

[22] L.J. Romans, *Massive $N = 2A$ supergravity in ten-dimensions*, *Phys. Lett.* **B 169** (1986) 374.

[23] E. Bergshoeff, M. de Roo, M.B. Green, G. Papadopoulos and P.K. Townsend, *Duality of type-II 7-branes and 8-branes*, *Nucl. Phys.* **B 470** (1996) 113 [hep-th/9601150].

[24] E. Bergshoeff, H.J. Boonstra and T. Ortin, *S-duality and dyonic p-brane solutions in type-II string theory*, *Phys. Rev.* **D 53** (1996) 7206 [hep-th/9508091].

[25] M.J. Duff, R.R. Khuri and J.X. Lu, *String solitons*, *Phys. Rep.* **259** (1995) 213 [hep-th/9412184];
G.T. Horowitz and A. Strominger, *Black strings and p-branes*, *Nucl. Phys.* **B 360** (1991) 197.

[26] J.C. Breckenridge, G. Michaud and R.C. Myers, *More D-brane bound states*, *Phys. Rev.* **D 55** (1997) 6438 [hep-th/9611174].

[27] J.H. Schwarz, *An* $\mathrm{SL}(2,\mathbb{Z})$ *multiplet of type-IIB superstrings*, *Phys. Lett.* **B 360** (1995) 13 [hep-th/9508143]; Erratum *ibid.* **364** (1995) 252.

[28] A.A. Tseytlin, *Self-duality of Born-Infeld action and Dirichlet 3-brane of type-IIB superstring theory*, *Nucl. Phys.* **B 469** (1996) 51 [hep-th/9602064].

[29] T.H. Buscher, *Quantum corrections and extended supersymmetry in new sigma models*, *Phys. Lett.* **B 159** (1985) 127; *A symmetry of the string background field equations*, *Phys. Lett.* **B 194** (1987) 59; *Path integral derivation of quantum duality in nonlinear sigma models*, *Phys. Lett.* **B 201** (1988) 466.

[30] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*, *Phys. Rep.* **244** (1994) 77 [hep-th/9401139].

[31] M. Dine, P. Huet and N. Seiberg, *Large and small radius in string theory*, *Nucl. Phys.* **B 322** (1989) 301.

[32] P. Meessen and T. Ortin, *An* $\mathrm{SL}(2,\mathbb{Z})$ *multiplet of nine-dimensional type-II supergravity theories*, *Nucl. Phys.* **B 541** (1999) 195 [hep-th/9806120].

[33] I. Washington Taylor, *D-brane field theory on compact spaces*, *Phys. Lett.* **B 394** (1997) 283 [hep-th/9611042].

[34] I.W. Taylor, *Lectures on D-branes, gauge theory and m(atrices)*, at 2nd Trieste Conference on Duality in String Theory, Trieste, Italy, 16-20 Jun 1997, hep-th/9801182.

[35] E. Bergshoeff and M.D. Roo, *D-branes and T-duality*, *Phys. Lett.* **B 380** (1996) 265 [hep-th/9603123];
E. Alvarez, J.L. F. Barbon and J. Borlaf, *T-duality for open strings*, *Nucl. Phys.* **B 479** (1996) 218 [hep-th/9603089].

[36] C. Bachas, *D-brane dynamics*, *Phys. Lett.* **B 374** (1996) 37 [hep-th/9511043];
J. Borlaf and Y. Lozano, *Aspects of T-duality in open strings*, *Nucl. Phys.* **B 480** (1996) 239 [hep-th/9607051];
M.B. Green, C.M. Hull and P.K. Townsend, *D-brane Wess-Zumino actions, t-duality and the cosmological constant*, *Phys. Lett.* **B 382** (1996) 65 [hep-th/9604119];
H. Dorn, *The mass term in non-abelian gauge field dynamics on matrix D-branes and T-duality in the sigma-model approach*, *J. High Energy Phys.* **04** (1998) 013 [hep-th/9804065];
J. Simon, *T-duality and effective D-brane actions*, hep-th/9812095.

[37] E. Alvarez, L. Alvarez-Gaume, J.L.F. Barbon and Y. Lozano, *Some global aspects of duality in string theory*, *Nucl. Phys.* **B 415** (1994) 71 [hep-th/9309039].

[38] A. Hashimoto and I.W. Taylor, *Fluctuation spectra of tilted and intersecting D-branes from the Born-Infeld action*, *Nucl. Phys.* **B 503** (1997) 193 [hep-th/9703217].

[39] P. Bain, *On the non-abelian Born-Infeld action*, hep-th/9909154.

[40] C.P. Bachas and P. Bain, unpublished;
A. Hashimoto and R.C. Myers, unpublished.

[41] A. Hashimoto, *The shape of branes pulled by strings*, *Phys. Rev.* **D 57** (1998) 6441 [hep-th/9711097];
D. Bak, J. Lee and H. Min, *Dynamics of BPS states in the Dirac-Born-Infeld theory*, *Phys. Rev.* **D 59** (1999) 045011 [hep-th/9806149].

[42] L. Thorlacius, *Born-Infeld string as a boundary conformal field theory*, *Phys. Rev. Lett.* **80** (1998) 1588 [hep-th/9710181].

[43] M. Fukuma, T. Oota and H. Tanaka, *Comments on T-dualities of Ramond-Ramond potentials on tori*, hep-th/9907132;
D. Brace, B. Morariu and B. Zumino, *T-duality and Ramond-Ramond backgrounds in the matrix model*, *Nucl. Phys.* **B 549** (1999) 181 [hep-th/9811213].

[44] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [hep-th/9610043].

[45] T. Banks, N. Seiberg and S. Shenker, *Branes from matrices*, *Nucl. Phys.* **B 490** (1997) 91 [hep-th/9612157].

[46] A.A. Tseytlin, *Vector field effective action in the open superstring theory*, *Nucl. Phys.* **B 276** (1986) 391;
D.J. Gross and E. Witten, *Superstring modifications of Einstein's equations*, *Nucl. Phys.* **B 277** (1986) 1.

[47] D. Kabat and I.W. Taylor, *Linearized supergravity from matrix theory*, *Phys. Lett.* **B 426** (1998) 297 [hep-th/9712185].

[48] I.W. Taylor and M.V. Raamsdonk, *Angular momentum and long-range gravitational interactions in matrix theory*, *Nucl. Phys.* **B 532** (1998) 227 [hep-th/9712159];
M.V. Raamsdonk, *Conservation of supergravity currents from matrix theory*, *Nucl. Phys.* **B 542** (1999) 262 [hep-th/9803003];
I.W. Taylor and M.V. Raamsdonk, *Supergravity currents and linearized interactions for matrix theory configurations with fermionic backgrounds*, *J. High Energy Phys.* **04** (1999) 013 [hep-th/9812239].

[49] M.R. Garousi and R.C. Myers, *Superstring scattering from D-branes*, *Nucl. Phys.* **B 475** (1996) 193 [hep-th/9603194];
A. Hashimoto and I.R. Klebanov, *Scattering of strings from D-branes*, *Nucl. Phys.* **55B** *(Proc. Suppl.)* (1997) 118 [hep-th/9611214]; *Decay of excited D-branes*, *Phys. Lett.* **B 381** (1996) 437 [hep-th/9604065];
S.S. Gubser, A. Hashimoto, I.R. Klebanov and J.M. Maldacena, *Gravitational lensing by p-branes*, *Nucl. Phys.* **B 472** (1996) 231 [hep-th/9601057];
I.R. Klebanov and L. Thorlacius, *The size of p-branes*, *Phys. Lett.* **B 371** (1996) 51 [hep-th/9510200].

[50] N. Seiberg, *Why is the matrix model correct?*, *Phys. Rev. Lett.* **79** (1997) 3577 [hep-th/9710009];
A. Sen, *D0-branes on* $\mathbb{T}^n$ *and matrix theory*, *Adv. Theor. Math. Phys.* **2** (1998) 51 [hep-th/9709220].

[51] I.W. Taylor and M.V. Raamsdonk, *Multiple Dp-branes in weak background fields*, hep-th/9910052.

[52] A.A. Tseytlin, *'No-force' condition and BPS combinations of p-branes in 11 and 10 dimensions*, *Nucl. Phys.* **B 487** (1997) 141 [hep-th/9609212].

[53] J. Madore, *The fuzzy sphere*, *Class. and Quant. Grav.* **9** (1992) 69; *Fuzzy physics*, *Ann. Phys. (NY)* **219** (1992) 187; *Quantum mechanics on a fuzzy sphere*, *Phys. Lett.* **B 263** (1991) 245; J. Hoppe, MIT Ph.D. thesis, 1982; *Elem. Part. Res. J.* (Kyoto) **80** (1989) 145.

[54] B. de Wit, J. Hoppe and H. Nicolai, *On the quantum mechanics of supermembranes*, *Nucl. Phys.* **B 305** (1988) 545.

[55] D. Kabat and I. Washington Taylor, *Spherical membranes in matrix theory*, *Adv. Theor. Math. Phys.* **2** (1998) 181 [hep-th/9711078];
S.-J. Rey, *Gravitating m(atrix) q-balls*, hep-th/9711081.

[56] M. Aganagic, C. Popescu and J.H. Schwarz, *D-brane actions with local kappa symmetry*, *Phys. Lett.* **B 393** (1997) 311 [hep-th/9610249]; *Gauge-invariant and gauge-fixed D-brane actions*, *Nucl. Phys.* **B 495** (1997) 99 [hep-th/9612080];
M. Cederwall, A. von Gussich, B.E. W. Nilsson, P. Sundell and A. Westerberg, *The dirichlet super-p-branes in ten-dimensional type-IIA and IIB supergravity*, *Nucl. Phys.* **B 490** (1997) 179 [hep-th/9611159];

E. Bergshoeff and P.K. Townsend, *Super D-branes*, *Nucl. Phys.* **B 490** (1997) 145 [hep-th/9611173];
M. Hatsuda and K. Kamimura, *Wess-zumino actions for IIA D-branes and their supersymmetries*, *Nucl. Phys.* **B 535** (1998) 499 [hep-th/9804087];
P.S. Howe, O. Raetzel and E. Sezgin, *On brane actions and superembeddings*, *J. High Energy Phys.* **08** (1998) 011 [hep-th/9804051];
M. Roček and A.A. Tseytlin, *Partial breaking of global $D = 4$ supersymmetry, constrained superfields and 3-brane actions*, *Phys. Rev.* **D 59** (1999) 106001 [hep-th/9811232];
S. Ryang, *Covariant gauge fixing for super IIA D-branes*, *Phys. Lett.* **B 451** (1999) 46 [hep-th/9809088];
I. Bandos et al., *On the equivalence of different formulations of the M-theory five-brane*, *Phys. Lett.* **B 408** (1997) 135 [hep-th/9703127];
I. Bandos et al., *Covariant action for the super-five-brane of M-theory*, *Phys. Rev. Lett.* **78** (1997) 4332 [hep-th/9701149];
T. Kimura and I. Oda, *Duality of super D-brane actions in type-II supergravity background*, *Nucl. Phys.* **B 551** (1999) 183 [hep-th/9811134];
T. Kimura and I. Oda, *Duality of super D-brane actions in general type-II supergravity background*, hep-th/9904019;
D. Sorokin, *Superbranes and superembeddings*, hep-th/9906142;
A. Westerberg and N. Wyllard, *Supersymmetric brane actions from interpolating dualisations*, *Nucl. Phys.* **B 560** (1999) 683 [hep-th/9904117].

[57] C.P. Bachas, P. Bain and M.B. Green, *Curvature terms in D-brane actions and their M-theory origin*, *J. High Energy Phys.* **05** (1999) 011 [hep-th/9903210].

[58] M.B. Green, J.A. Harvey and G. Moore, *I-brane inflow and anomalous couplings on D-branes*, *Class. and Quant. Grav.* **14** (1997) 47 [hep-th/9605033];
Y.-K.E. Cheung and Z. Yin, *Anomalies, branes and currents*, *Nucl. Phys.* **B 517** (1998) 69 [hep-th/9710206].

[59] K. Dasgupta, D.P. Jatkar and S. Mukhi, *Gravitational couplings and $\mathbb{Z}_2$ orientifolds*, *Nucl. Phys.* **B 523** (1998) 465 [hep-th/9707224];
K. Dasgupta and S. Mukhi, *Anomaly inflow on orientifold planes*, *J. High Energy Phys.* **03** (1998) 004 [hep-th/9709219];
B. Stefanski, *Gravitational couplings of D-branes and O-planes*, *Nucl. Phys.* **B 548** (1999) 275 [hep-th/9812088];
B. Craps and F. Roose, *Anomalous D-brane and orientifold couplings from the boundary state*, *Phys. Lett.* **B 445** (1998) 150 [hep-th/9808074];
J.F. Morales, C.A. Scrucca and M. Serone, *Anomalous couplings for D-branes and O-planes*, *Nucl. Phys.* **B 552** (1999) 291 [hep-th/9812071];
B. Craps and F. Roose, *(Non-)anomalous D-brane and O-plane couplings: the normal bundle*, *Phys. Lett.* **B 450** (1999) 358 [hep-th/9812149];
C.A. Scrucca and M. Serone, *Anomalies and inflow on D-branes and O-planes*, *Nucl. Phys.* **B 556** (1199) 197 [hep-th/9903145].

[60] J. Scherk and J.H. Schwarz, *Dual models and the geometry of space-time*, *Phys. Lett.* **B 52** (1974) 347;
S. Mukhi, *Finiteness of nonlinear sigma models with parallelizing torsion*, *Phys. Lett.* **B 162** (1985) 345;
The geometric background field method, renormalization and the Wess-Zumino term in nonlinear sigma models, *Nucl. Phys.* **B 264** (1986) 640.

[61] M.B. Green, J.H. Schwarz and E. Witten, *Superstring theory,* vol. 2, Cambridge University Press, 1987.

[62] O.J. Ganor, S. Ramgoolam and I.W. Taylor, *Branes, fluxes and duality in m(atrix)-theory*, *Nucl. Phys.* **B 492** (1997) 191 [hep-th/9611202].

[63] J.G. Russo and A.A. Tseytlin, *Waves, boosted branes and BPS states in M-theory*, *Nucl. Phys.* **B 490** (1997) 121 [hep-th/9611047].

[64] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[65] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162];
M.R. Douglas and C. Hull, *D-branes and the noncommutative torus*, *J. High Energy Phys.* **02** (1998) 008 [hep-th/9711165].

[66] N. Constable, O. Tafjord and R.C. Myers, in preparation.

JHEP12(1999)022